CURVED INFINITY-ALGEBRAS AND THEIR CHARACTERISTIC CLASSES

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Abstract. In this paper we study a natural extension of Kontsevich's characteristic class construction for A_∞- and L_∞-algebras to the case of curved algebras. These define homology classes on a variant of his graph homology which allows vertices of valence ≥ 1. We compute this graph homology, which is governed by star-shaped graphs with odd-valence vertices. We also classify nontrivially curved cyclic A_∞- and L_∞- algebras over a field up to gauge equivalence, and show that these are essentially reduced to algebras of dimension at most two with only even-ary operations. We apply the reasoning to compute stability maps for the homology of Lie algebras of formal vector fields. Finally, we explain a generalization of these results to other types of algebras, using the language of operads. A key observation is that operads governing curved algebras are closely related to the Koszul dual of those governing unital algebras.

Contents

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1. Introduction

Kontsevich has associated certain characteristic classes to finite-dimensional L_∞- or A_∞-algebras equipped with an invariant inner product in [Kon93, Kon94]. These are expressed in terms of the homology of certain complexes spanned by graphs with some additional structures. This construction is by now well-understood both from the point of view of Lie algebra homology and topological conformal field theory; see, for example, [HL08].

In this note, we explore a natural generalization of this construction to the case of *curved* algebras, introduced by Positselski in [Pos93]. It turns out that a complete description of these classes, and of the homology of the associated graph complexes, is possible. We show that these are all obtainable from one-dimensional algebras, and that these classes are zero for algebras with zero curvature. This contrasts with the corresponding problem for conventional graph complexes, which is still widely open.

Roughly, this holds for the following reason. The usual (uncurved) L_∞-algebras correspond to odd vector fields X on the formal neighborhood of zero in a vector space satisfying $[X, X] = 0$, which have a critical point at the origin. Equivalence classes of these are complicated. On the other hand, nontrivially curved L_∞-algebras correspond to the same vector fields except having no critical points, and these can all be put in a standard form. A similar result extends to the noncommutative setting for A_∞-algebras. Cyclic algebras replace vector fields with Hamiltonian vector fields, in the context of the formal neighborhood of zero in a symplectic vector space. In this case, the standard form is somewhat less trivial. Our first results, Theorems 2.1 and 2.8, classify all of these vector fields (i.e., types of algebras) up to gauge equivalence. In particular, in the ordinary (noncyclic) case, nontrivially curved A_∞- or L_∞-algebras are gauge equivalent (i.e., homotopy isomorphic) to the algebra with the same curvature and zero multiplication; in the cyclic case, algebras are gauge equivalent to the direct sum of a curved algebra of dimension at most two of a certain form (but having nontrivial multiplications in general), with an algebra having the trivial infinity-structure.

As explained by Kontsevich, the aforementioned graph complexes can be viewed as computing the stable homology of Lie algebras of symplectic vector fields on a vector space W (in the A_∞ case, one should take *noncommutative* symplectic vector fields). This motivates us to consider the stability maps. In this direction, we prove that the map from the Lie algebra of symplectic vector fields on W vanishing at the origin to the homology of the Lie algebra of *all* vector fields on $W \oplus \mathbb{C} \cdot w$, where w is an odd vector, is zero. Similarly, we show the same for the Lie algebra of noncommutative symplectic vector fields.

The precise relation to the previous result is as follows. Any cyclic L_∞-algebra structure on V defines an unstable characteristic class in the homology of the Lie algebra of symplectic vector fields on the shifted vector space $W = \Pi V$. As $\dim V \to \infty$, the homology of this Lie algebra converges to the graph homology (at least if V is only growing in the purely even or purely odd direction: see Theorem 3.12 below), and the image of the unstable characteristic class under the stability maps gives, in the limit, the aforementioned (stable) characteristic class. Hence, our result above says that the unstable curved characteristic class of an algebra with zero curvature already maps to zero under the first stability map $W := \Pi V \hookrightarrow W \oplus \mathbb{C} \cdot w$.

These results hint at a triviality of curved infinity-algebras from a homological point of view, at least when the cyclic structure is not considered. A similar result on the triviality

of the corresponding derived categories was obtained recently in [KLN10]; another manifestation of this triviality principle is briefly discussed in the last section of this paper.

Finally, we generalize these results to the operadic setting, i.e., to types of algebras other than associative and Lie algebras. A key point is that operads governing curved algebras are closely related to the Koszul dual of operads governing unital algebras. Namely, a curved version of an operad is obtained by the cobar construction from a unital version of the Koszul dual operad. The presence of the unit provides a contracting homotopy on large summands of the obtained graph complexes.

In particular, the operadic generalization applies to Poisson, Gerstenhaber, BV, permutation, and pre-Lie algebras. For the most part, the generalization is straightforward, and we restrict ourselves with giving only an outline of arguments in this section. There is, however, one important aspect which is less visible in the special cases of commutative and ribbon graphs: a curved graph complex associated with a cyclic (or even modular) operad $\mathcal{O}$ is quasi-isomorphic to a variant of the *deformation complex of a curved $\mathcal{O}$-algebra on a one-dimensional space*. Therefore, this graph complex supports the structure of a differential graded Lie algebra. This differential Lie algebra, and its Chevalley-Eilenberg complex, appeared in various guises in the works of Zwiebach-Sen, Costello and Harrelson-Voronov-Zuniga on quantum master equation, [SZ96, Cos09, HVZ07].

1.1. **Notation and conventions.** In this paper we work in the category of $\mathbb{Z}/2$-graded vector spaces (also known as (super)vector spaces) over $\mathbb{C}$ although all results continue to hold in the $\mathbb{Z}$-graded context and when $\mathbb{C}$ is replaced by any field of characteristic zero. We will usually refer to these graded vector spaces simply as "spaces." The parity of a homogeneous vector v in a space will be denoted by $|v|$. The adjective 'differential graded' will mean 'differential $\mathbb{Z}/2$-graded' and will be abbreviated as 'dg'. A (commutative) differential graded (Lie) algebra will be abbreviated as (c)dg(l)a. All of our unmarked tensors are understood to be taken over $\mathbb{C}$. For a $\mathbb{Z}/2$-graded vector space $V = V_0 \oplus V_1$ the symbol ΠV will denote the *parity reversion* of V; thus $(\Pi V)_0 = V_1$ while $(\Pi V)_1 = V_0$.

We will make use of the language of *formal*[1] spaces and algebras (which exist since [Lef42] under the name "linearly compact"; see, e.g., [HL09] for a recent treatment relevant to the present work, under the present name). A formal space is an inverse limit of finite-dimensional spaces. The functor of taking the linear dual establishes an anti-equivalence between the category of (discrete) vector spaces and that of formal vector spaces.

It will always be clear from the context whether we work with formal or discrete vector spaces, and we will typically not mention this specifically later on; the tensor product of two formal spaces is understood to be their *completed* tensor product. Furthermore, the symbol V will be reserved for a discrete space, with its dual V^* therefore a formal space.

In particular, we will work with formal (c)dg(l)as. The main examples will be completed tensor and symmetric algebras on formal spaces W; these will be denoted by $\hat{T}W$ and $\hat{S}W$ respectively. Note that we will *never* consider the uncompleted TW and SW when W is formal, and similarly never consider the completed $\hat{T}V$ or $\hat{S}V$ when V is discrete, so as to stay in either the category of formal spaces or that of discrete spaces.

[1]Here the word "formal" is understood in the sense of a formal neighborhood, which differs from the notion of "formality" in rational homotopy theory.

The Lie algebras of continuous derivations of $\hat{T}W$ and of $\hat{S}W$ will be denoted by $\mathrm{Der}(\hat{T}W)$ and $\mathrm{Der}(\hat{S}W)$ respectively; we will also consider their Lie subalgebras $\mathrm{Der}^0(\hat{T}W)$ and $\mathrm{Der}^0(\hat{S}W)$ consisting of derivations having no constant terms.

We mention here two potential pitfalls present in this framework. Firstly, the categories of discrete and formal vector spaces are not disjoint: the spaces which are both discrete and formal are precisely finite-dimensional spaces. And secondly, not every space is either discrete or formal; moreover such spaces arise as a result of some natural operations with discrete or formal spaces. For example if U and W are infinite-dimensional formal or discrete spaces then the vector space $\mathrm{Hom}(U, W)$ is neither formal nor discrete. Similarly, the Lie algebras $\mathrm{Der}(\hat{T}V^*)$ and $\mathrm{Der}(TV)$ will be neither formal nor discrete if V is infinite-dimensional.

Therefore, to avoid possible confusion, we make the blanket assumption that the $\mathbb{Z}/2$-graded vector space V which appears throughout the paper, in addition to being discrete as above, is in fact *finite-dimensional*. This way all the objects we consider will live in either the category of formal spaces or the category of discrete spaces (but not both: so each finite-dimensional space we consider will be viewed in only one way). The price we pay is that some of our results are not formulated in maximal generality; namely Theorem 2.1 and Claim 5.8 do not need the space V to be finite-dimensional (although essentially none of the exposition needs to be modified to obtain this generalization).

For a *formal* dgla $\mathfrak{g}$ its Chevalley-Eilenberg cohomological complex will be denoted by $\mathrm{CE}^\bullet(\mathfrak{g})$. This is defined as follows (note that the definition *differs* from the usual one in where completions are taken, since $\mathfrak{g}$ is formal rather than discrete): the underlying graded vector space of $\mathrm{CE}^\bullet(\mathfrak{g})$ is $S\Pi\mathfrak{g}^*$ and the differential is given as a sum of two maps d_{I} and d_{CE}. Here d_{I} and d_{CE} are both specified by their restriction onto $\Pi\mathfrak{g}^*$ and extended to the whole $S\Pi\mathfrak{g}^*$ by the Leibniz rule; further $d_{\mathrm{I}} : \Pi\mathfrak{g}^* \to \Pi\mathfrak{g}^*$ is the shift of the dual of the internal differential on $\mathfrak{g}$ whereas $d_{\mathrm{CE}} : \Pi\mathfrak{g}^* \to S^2(\Pi\mathfrak{g}^*)$ is induced by the commutator map $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$.

This is in general a $\mathbb{Z}/2$-graded complex. In the case that the differential d is zero, it has an additional grading by *cohomological degree*, i.e., $S^i\Pi\mathfrak{g}^*$ is in degree i. The corresponding homological complex is the linear dual: $\mathrm{CE}_\bullet(\mathfrak{g}) = (\mathrm{CE}^\bullet(\mathfrak{g}))^*$, which has the underlying formal space $\hat{S}\Pi\mathfrak{g}$. Note that in some papers (e.g. [HL08]) the Chevalley-Eilenberg complex of a graded Lie algebra $\mathfrak{g}$ is defined using the (in our case completed) *exterior* algebra $\hat{\Lambda}\mathfrak{g}$; this definition is equivalent to ours under a canonical isomorphism $\hat{S}(\Pi\mathfrak{g}) \cong \hat{\Lambda}\mathfrak{g}$ where

$$\Pi g_1 \ldots \Pi g_n \mapsto (-1)^{|g_n|+2|g_{n-1}|+\ldots+(n-1)|g_1|} g_1 \wedge \ldots \wedge g_n.$$

2. Gauge equivalence classes of curved (cyclic) A_∞- and L_∞-algebras

2.1. **Curved A_∞- and L_∞- algebras.** We recall the definition of A_∞- and L_∞-algebras following [HL08], as well as their curved analogues; cf. e.g., [Nic08] and references therein.

A curved A_∞-algebra structure on a (finite-dimensional[2]) space V is a continuous odd derivation m of the formal dga $\hat{T}\Pi V^*$ and a curved L_∞-algebra structure on V is a continuous odd derivation m of the formal cdga $\hat{S}\Pi V^*$; additionally m is required to square to zero in both cases. An ordinary (i.e. uncurved) A_∞- or L_∞-structure is specified by the requirement

[2]As in §1.1, V is considered as a discrete space throughout; in the present subsection one could allow it to be infinite-dimensional and discrete. We will not make further mention of this.

that m have no constant term. The components $m_i : T^i\Pi V \to \Pi V$ or $S^i\Pi V \to \Pi V$ of the dual of m are the structure maps of the corresponding A_∞- or L_∞-structure.

We note that sometimes it is convenient to use the more traditional way of writing the structure maps of an A_∞- or L_∞-algebra V as $T^iV \to V$ or $\Lambda^iV \to V$; these maps will then be even or odd depending on whether i is even or odd. To alleviate the notation we will still write m_i for these maps when the meaning is clear from the context.

We are interested in *gauge equivalence* classes of A_∞- or L_∞- structures on a fixed space V. In particular, this equivalence relation implies other relations found in the literature under the names of homotopy or quasi-isomorphism.

Namely, a gauge equivalence between (V, m) and (V, m') is defined as a derivation $\xi \in \mathrm{Der}^0(\hat{T}\Pi V^*)$ or $\xi \in \mathrm{Der}^0(\hat{S}\Pi V^*)$ which is even and satisfies $m' = e^{\mathrm{ad}\,\xi}m$.[3] In particular, such a gauge equivalence yields an isomorphism of dgas $e^\xi : (\hat{T}\Pi V^*, m) \xrightarrow{\sim} (\hat{T}\Pi V^*, m')$ or cdgas $e^\xi : (\hat{S}\Pi V^*, m) \xrightarrow{\sim} (\hat{S}\Pi V^*, m')$, and we usually denote the gauge equivalence by e^ξ.

Theorem 2.1. If (V, m) is a curved A_∞- or L_∞- algebra for which the curvature $m_0 = c \in V_0$ is nonzero, then m is gauge equivalent to the structure $m' = c$ with all higher multiplications $m'_i = 0$ for $i > 0$.

Roughly, the above is saying that, when an (odd, noncommutative) formal vector field is nonzero evaluated at zero, then it is equivalent to a constant vector field up to (generally nonlinear) change of coordinates.

As a corollary of the theorem, it follows that any two nontrivially curved algebras with the same underlying graded vector space V are gauge equivalent.

Proof of Theorem 2.1. We consider the A_∞ case; the L_∞ case is similar.

Any A_∞-algebra structure (V, m') with $m'_0 = m_0 = c$ can be viewed as a deformation of (V, m_0). Indeed, let us introduce a formal parameter $\hbar$; then (V, m') is equivalent to the deformed structure $(V, m'_\hbar)$, where $m'_\hbar = m'_0 + \sum_{i \geq 1} \hbar^i m'_i$. This yields an equivalence with deformed structures whose i-ary operations are homogeneous of degree i in $\hbar$.

The structures of the form (V, m') are governed by the dgla $\mathrm{Der}(\hat{T}\Pi V^*, [, c])$, where $c \in V$ is viewed as an odd constant derivation of $\hat{T}\Pi V^*$. Formal deformations (such as $(V, m'_\hbar)$) are governed by the dgla $\mathrm{Der}(\hat{T}\Pi V^*[[\hbar]], [, c])$. Gauge equivalences e^ξ of Maurer-Cartan elements of $\mathrm{Der}(\hat{T}\Pi V^*, [, c])$, which we can assume satisfy $\xi_0 = 0 = \xi_1$ (so as to not change the curvature) are identified with gauge equivalences $e^{\frac{1}{\hbar}\xi_\hbar}$ (for $\xi_\hbar := \sum_i \hbar^i \xi_i$) of the corresponding Maurer-Cartan elements of $\mathrm{Der}(\hat{T}\Pi V^*[[\hbar]], [, c])$, which preserve the grading $|\hbar| = 1 = |V^*|$.

Since gauge equivalence classes of deformations of Maurer-Cartan elements are preserved under quasi-isomorphisms of dglas, it suffices to show that $\mathrm{Der}(\hat{T}\Pi V^*, [, c])$ is acyclic. Let us write down this complex $C^\bullet$ explicitly; note that $C^\bullet$ is a version of the Hochschild complex in the curved setting.

Set $C^i := \mathrm{Hom}((\Pi V)^{\otimes i}, \Pi V)$ with the differential $d : C^i \to C^{i-1}$ given by the formula, for $f \in C^i$, and $x_1, \ldots, x_{i-1} \in \Pi V$:

$$df(x_1, \ldots, x_{i-1}) = \sum_k (-1)^{|x_1| + \cdots + |x_k|} f(x_1, \ldots, x_k, c, x_{k+1}, \ldots, x_{i-1}).$$

[3]In the case that our ground field is not $\mathbb{C}$, e^ξ still makes sense if we require additionally that $\xi_1 = 0$ as well, i.e., ξ has no linear term. We can then modify the above by saying that a gauge equivalence is a composition of such an equivalence (for $\xi_1 = 0$) with a linear isomorphism of V.

We construct an explicit contracting homotopy. Choose an odd linear map $\epsilon : \Pi V \to \mathbb{C}$ such that $\epsilon(c) = 1$ (here, odd means that $\epsilon|_{\Pi(V_1)=(\Pi V)_0} = 0$). Define maps $s_i : C^i \to C^{i+1}$ by the formula, for $f \in C^i$:

$$s_i f(x_1, \ldots, x_{i+1}) = \epsilon(x_1) f(x_2, \ldots, x_{i+1}). \tag{2.2}$$

Then,

$$ds_i f(x_1, \ldots, x_i) + s_{i-1} df(x_1, \ldots, x_i) = \epsilon(c) f(x_1, \ldots, x_i) = f(x_1, \ldots, x_i). \tag{2.3}$$

□

2.2. **Cyclic algebras.** We now extend the results of the previous subsection to the case of algebras with a cyclic inner product. By an *inner product* on a $\mathbb{Z}/2$-graded vector space V we mean a nondegenerate symmetric graded bilinear form $(-,-) : V \otimes V \to \mathbb{C}$, where $\mathbb{C}$ is considered to be in even degree, and V is required to be finite-dimensional. An *inner product space* is a space equipped with an inner product.

Definition 2.4. Let (V, m) be a finite-dimensional A_∞- or L_∞-algebra. A cyclic inner product on V is an inner product $(-,-) : V \otimes V \to \mathbb{C}$ for which the tensors $(m_i(v_1, \ldots, v_i), v_{i+1})$ are invariant with respect to the signed cyclic permutations of arguments:

$$(m_i(v_1, \ldots, v_i), v_{i+1}) = (-1)^{i+(|v_1|+\ldots+|v_i|)|v_{i+1}|}(m_i(v_{i+1}, v_1 \ldots, v_{i-1}), v_i)$$

We call an algebra (V, m) a *cyclic* A_∞- or L_∞-algebra when it is equipped with such a cyclic inner product.

We need to define a subspace of $\mathrm{Der}(\hat{T}\Pi V^*)$ of *cyclic* derivations. To do so, first note that $\mathrm{Der}(\hat{T}\Pi V^*) \cong \mathrm{Hom}(\Pi V^*, \hat{T}\Pi V^*)$ via the restriction map. Using the inner product, we can identify the latter with $(\Pi V^* \otimes \hat{T}\Pi V^*)$. Next, on any tensor power $(\Pi V^*)^{\otimes n}$, we can define the (graded) cyclic permutation operator $\sigma : (\Pi V^*)^{\otimes n} \to (\Pi V^*)^{\otimes n}$. This extends to a continuous linear automorphism of $(\Pi V^* \otimes \hat{T}\Pi V^*)$, and yields a continuous linear automorphism of $\mathrm{Der}(\hat{T}\Pi V^*)$.

Definition 2.5. Let $\mathrm{CDer}(\hat{T}\Pi V^*)$ denote the space of derivations which are invariant under cyclic permutation: we call these *cyclic* derivations. Similarly, define $\mathrm{CDer}^0(\hat{T}\Pi V^*)$ as those cyclic derivations with zero constant term (i.e., preserving the augmentation $\hat{T}^{\geq 1}\Pi V^*$). Finally, define the spaces $\mathrm{CDer}(\hat{S}\Pi V^*)$ and $\mathrm{CDer}^0(\hat{S}\Pi V^*)$ analogously.

Note that $\mathrm{CDer}(\hat{S}\Pi V^*)$ can be viewed as the subspace of $\mathrm{CDer}(\hat{T}\Pi V^*)$ landing in symmetric tensors, and similarly for the version with zero constant term.

Given V together with an inner product, cyclic curved A_∞-structures are the same as odd derivations $\xi \in \mathrm{CDer}(\hat{T}\Pi V^*)$ which square to zero. Similarly, uncurved structures correspond to odd square-zero $\xi \in \mathrm{CDer}^0(\hat{T}\Pi V^*)$, and the L_∞-versions are obtained by replacing $\hat{T}$ with $\hat{S}$. See, e.g., [HL09] for details.

Definition 2.6. A gauge equivalence of cyclic A_∞- or L_∞- structures (V, m) and (V, m') on a fixed inner product space V is a map e^ξ where $\xi \in \mathrm{CDer}^0(\hat{T}\Pi V^*)$ or $\xi \in \mathrm{CDer}^0(\hat{S}\Pi V^*)$ satisfies $m' = e^{\mathrm{ad}\,\xi} m$.

Remark 2.7. In the literature, the term *symplectic* is sometimes used instead of cyclic, the idea being that an inner product on V is equivalent to a (constant) symplectic structure on ΠV, so that the cyclic derivations on $\hat{S}\Pi V^*$ or $\hat{T}\Pi V^*$ are the same as formal (possibly

noncommutative) symplectic vector fields on ΠV; a cyclic gauge equivalence could then be interpreted as a (formal, noncommutative) symplectomorphism (preserving the $\mathbb{Z}/2$-grading). Note that it follows from this interpretation that the cyclic derivations of V form a Lie superalgebra, and the cyclic gauge equivalences a Lie group.

Theorem 2.8.

(a) If (V,m) is a curved A_∞-algebra with a cyclic inner product for which the curvature $m_0 = c \in V_0$ is nonzero, and $c' \in V$ any even element for which $(c,c') = 1$, then (V,m) is gauge equivalent to the structure m' with $m'_0 = c$, and higher multiplications

$$m'_{2i-1} = 0, i \geq 1, \tag{2.9}$$

$$m'_{2i}(x_1, \ldots, x_{2i}) = \Big(\prod_{j=1}^{2i}(c', x_j)\Big) \cdot (m_{2i}(c, c, \ldots, c), c) \cdot c'. \tag{2.10}$$

(b) If (V,m) is a curved L_∞-algebra with a cyclic inner product for which the curvature $m_0 = c \in V_0$ is nonzero, then (V,m) is gauge equivalent to the structure m' with $m'_0 = c$, and higher multiplications $m'_i = 0$ zero for all $i \geq 1$.

One immediately deduces

Corollary 2.11.

(a) Two curved cyclic A_∞-algebra structures (V,m) and (V,m') on the same underlying inner product space V with nonzero curvature m_0, m'_0 are gauge equivalent if and only if

$$(m_0, m_0) = (m'_0, m'_0), \quad (m_{2i}(m_0, \ldots, m_0), m_0) = (m'_{2i}(m'_0, \ldots, m'_0), m'_0), \forall i \geq 1. \tag{2.12}$$

(b) Two curved cyclic L_∞-algebra structures (V,m) and (V,m') on the same underlying inner product space V with nonzero curvature m_0, m'_0 are gauge equivalent if and only if $(m_0, m_0) = (m'_0, m'_0)$.

Remark 2.13. Another, perhaps intuitively more clear, way to understand this result is as follows. Consider the curved cyclic A_∞-algebra V with curvature c whose underlying inner product space is spanned by a single even vector c with $(c,c) = 1$ and higher products $m_{2i}(c, \ldots, c) = t_i c$; $m_{2i+1} = 0$ for $i = 0, 1, \ldots$. Here t_i are arbitrary numbers.

Consider also the curved cyclic A_∞-algebra V' whose underlying space is spanned by two even vectors c and c' with $(c,c') = (c',c) = 1, (c,c) = (c',c') = 0$. The A_∞-structure is given as $m_0 = c$ and, for $i > 0$, $m_{2i}(c, \ldots, c) = t_i c'$, where t_i are arbitrary, and all other higher products are zero.

Then any cyclic curved A_∞-algebra with nonzero curvature is gauge equivalent to the direct sum of either V or V' with an A_∞-algebra having zero A_∞-structure.

Proof of Theorem 2.8. (a) We modify the previous obstruction theory argument. Deformations of the algebra with $m_0 = c$ and all higher operations zero are governed by the subcomplex

$$B^\bullet := \mathrm{CDer}(\hat{T}\Pi V^*, [, c]) \subset C^\bullet \tag{2.14}$$

of the one considered in Theorem 2.1.

In Lemma 2.17 below, we show that $B^\bullet$ is quasi-isomorphic to the subcomplex $B_0^\bullet$ spanned by the cocycles

$$\epsilon^{i+1}(x_1,\ldots,x_i) := \epsilon(x_1)\cdots\epsilon(x_i)c', \quad \epsilon(v) := (c', v). \tag{2.15}$$

Note that, when i is odd, $\epsilon^{i+1} = 0$. Moreover, $B_0^\bullet$ is an abelian sub-dgla with zero differential.

Using the lemma, the result follows as in the proof of Theorem 2.1. In more detail, the gauge equivalence classes of Maurer-Cartan elements of $B^\bullet[[\hbar]]$ and $B_0^\bullet[[\hbar]]$ which are zero modulo $\hbar$ are identified. Moreover, the formal Maurer-Cartan elements which are homogeneous of the form $\sum_{i\geq 1}\hbar^i m_i$ for $m_i \in B^i$ or B_0^i are identified with actual Maurer-Cartan elements $\sum_{i\geq 1} m_i$ with zero constant term. Hence, the gauge equivalence classes of cyclic curved A_∞- structures with $m_0 = c$ are identified with gauge equivalence classes of Maurer-Cartan elements of $B_0^\bullet$ with zero constant term. Since the latter is abelian with zero differential, all elements are Maurer-Cartan, and define distinct gauge equivalence classes.

We deduce that all curved A_∞-structures with $m_0 = c$ are gauge equivalent to one with all odd operations equal to zero, and even operations given by some multiple of the operation

$$(x_1,\ldots,x_{2i}) \mapsto \Big(\prod_{j=1}^{2i}(c', x_j)\Big)\cdot c'. \tag{2.16}$$

It remains to show that this multiple is $(m_{2i}(c,c,\ldots,c),c)$. This follows by taking the component of the original operation in the direct summand $B_0^i \subseteq B^i = (B_0^i \oplus B_+^i)$, using the definition of B_+^i in (2.22) below. One must show that the result is gauge equivalent to the initial structure. To do so, one can inductively construct a gauge equivalence killing off the part in B_k^i below for $k \geq 1$ (first inductively on i, then reverse-inductively on k), parallel to the construction of the contracting homotopy in Lemma 2.17. We omit the details.

(b) We can apply the same argument as above, except now with $B := \mathrm{CDer}(\hat{S}\Pi V^*, [,c])$. The same argument as above applies and we deduce the same result, except that this time $m_{2i}(c,c,\ldots,c) = 0$ for all $i \geq 1$ by skew-symmetry of m_{2i}. □

Lemma 2.17. Keeping the assumptions and notation of the theorem, the complex $B^\bullet$ decomposes as $B^\bullet = B_0^\bullet \oplus B_+^\bullet$, where $B_+^\bullet$ is acyclic. Moreover, the inclusion $B_0^\bullet \hookrightarrow B^\bullet$ is a quasi-isomorphism of dglas.

Proof. The second statement follows from the first, since $B_0^\bullet \hookrightarrow B^\bullet$ is a dgla map.

To prove the first statement, we modify the contracting homotopy s_i from Theorem 2.1 to act on $B^\bullet$. The result will *not* be a contracting homotopy, but will instead accomplish the desired goal.

Define maps $s_i' : B^i \to B^{i+1}$ by

$$s_i' f = \sum_{j=0}^{i+1}\sigma^j(s_i f), \tag{2.18}$$

where σ^j is the j-th power of the cyclic permutation defined above Definition 2.5. We now compute $s'_{i-1}d + ds'_i$. Assume that $f \in B^i$. In this case, we may use the formula

$$s'_i f(x_1, \ldots, x_{i+1}) = \sum_{j=1}^{i+1} (-1)^{|x_1|+\ldots+|x_{j-1}|} \epsilon(x_j) f(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{i+1}) \\ + (-1)^{|x_1|+\ldots+|x_{i+1}|} (f(x_1, \ldots, x_i), x_{i+1}) c'. \tag{2.19}$$

Then, we compute

$$(s'_{i-1}d + ds'_i) f(x_1, \ldots, x_i) = (i+1)\epsilon(c) f(x_1, \ldots, x_i) \\ - \big(\sum_{j=1}^{i+1} \epsilon(x_j) f(x_1, \ldots, x_{j-1}, c, x_{j+1}, \ldots, x_i) + (f(x_1, \ldots, x_i), c) c' \big). \tag{2.20}$$

Thus, the operator on the RHS is contractible. Next, for $k \leq i+1$, define subspaces

$$B^i_k \subset B^i, \quad B^i_k = B^i \cap \big(\mathbb{C}[S_{i+1}] \cdot \big((\mathbb{C} \cdot \epsilon)^{\otimes(i+1-k)} \otimes ((\mathbb{C} \cdot c)^{\perp})^{\otimes k} \big) \big), \tag{2.21}$$

$$B^i_+ := \sum_{k=1}^{i+1} B^i_k. \tag{2.22}$$

Note that $B^\bullet_+$ is a subcomplex, and $B^\bullet = B^\bullet_0 \oplus B^\bullet_+$.

We claim that the RHS of (2.20) acts as $k \cdot \mathrm{Id}$ on B^i_k for all k. This follows directly. As a result, $s'_{i-1}d + ds'_i$ restricts to zero on $B^\bullet_0$ and to an automorphism on $B^\bullet_+$. This proves the lemma. $\square$

Remark 2.23. Lemma 2.17 is equivalent to the statement that the cyclic (co)homology of any curved A_∞-algebra V whose structure maps are all zero, except for m_0 (which is not zero), is isomorphic to the cyclic (co)homology of a one-dimensional A_∞-algebra V_1 having the same property. See [GJ90, HL09] for the notion of cyclic (co)homology of A_∞-algebras. These (co)homologies could be computed in other ways from the above, for instance, with the help of Connes' exact sequence connecting Hochschild (co)homology with cyclic (co)homology (which appears in [GJ90] in the curved setting), since Theorem 2.1 shows that the Hochschild (co)homologies of V and V_1 are trivial.

3. Characteristic classes of curved algebras

3.1. **The uncurved case.** We briefly recall Kontsevich's construction of characteristic classes of finite-dimensional A_∞- or L_∞-algebras with cyclic inner products. First, we recall the definition of certain graph complexes, for which cyclic A_∞- or L_∞-algebras will produce cycles.

Definition 3.1. A graph is a tuple $(H, V, E, \varphi_V, \varphi_E)$ of sets H, V, E of *half-edges*, *vertices*, and *edges*, and surjective maps $\varphi_V : H \to V, \varphi_E : H \to E$ such that the fibers of φ_E all have cardinality two.

Given $\Gamma = (H, V, E, \varphi_V, \varphi_E)$, we will also write $H_\Gamma = H, V_\Gamma = V, E_\Gamma = E, \varphi^\Gamma_V = \varphi_V$, and $\varphi^\Gamma_E = \varphi_E$.

Definition 3.2. A ribbon graph is a graph together with a cyclic ordering on each fiber $\varphi_V^{-1}(t)$.

Intuitively, one may think of the edges of ribbon graphs as slightly fattened, which explains the cyclic ordering at vertices.

Kontsevich's graph complexes have a basis of graphs of a certain type, with differential taking a graph to the sum over all edges of the contracted graph obtained by shrinking that edge to a point, together with a sign. To make this precise requires the notion of *orientation*:

Definition 3.3. An orientation on a graph is a choice of ordering of all the half-edges, ordering of all the vertices, and a sign ± 1, modulo the relation that applying a transposition to the ordering of either the half-edges or the vertices is the same as changing the sign.

Definition 3.4. An oriented graph is a graph equipped with an orientation. An isomorphism of oriented graphs is an isomorphism of graphs which preserves orientation. Similarly, the same definition applies replacing "graph" with "ribbon graph."

Next, given a graph $\Gamma = (H, V, E, \varphi_V, \varphi_E)$ and an edge $e \in E$ with endpoints $v_1, v_2 \in V$ meeting halves $h_1, h_2 \in H$, one defines the contracted graph $d_e(\Gamma) = (H, V/\{v_1 = v_2\}, E, \varphi'_V, \varphi_E)$ by identifying the endpoints v_1 and v_2. If Γ is moreover a ribbon graph, with the cyclically ordered sets $\varphi_V^{-1}(v_1) = (a_1, a_2, \ldots, a_i = h_1)$ and $\varphi_V^{-1}(v_2) = (b_1, b_2, \ldots, b_j = h_2)$, then the cyclic ordering of the half edges at the new vertex $v = v_1 = v_2$ is defined as $(a_1, a_2, \ldots, a_{i-1}, b_1, b_2, \ldots, b_{j-1})$. Finally, if Γ is equipped with an orientation, where the half-edges are ordered as $h_1, h_2, p_1, \ldots, p_m$ and with vertices ordered by $v_1, v_2, w_1, \ldots, w_k$, the new orientation is given by the ordering $p_1, \ldots, p_m$ and $v, w_1, \ldots, w_k$ of vertices, without changing the sign.

Consider the graded vector space with basis the isomorphism classes of oriented graphs *whose vertices have valence* ≥ 2, modulo the relation that a graph is negative its opposite orientation. The grading is given by the number of vertices. Let $\mathcal{G}$ be the completion of this graded vector space with respect to the number of edges (so *not* with respect to the defining grading on the vector space, which is by number of vertices). Similarly, define $\mathcal{G}_r$ using ribbon graphs rather than graphs. In other words (since we work over a field of characteristic zero, so that 2 is invertible), these are the spaces of possibly infinite linear combinations of isomorphism classes of oriented graphs which are not isomorphic to the graph obtained by reversing the orientation.

Then, it is a result of [Kon93] that

$$d(\Gamma) := \sum_{e \in E_\Gamma} d_e(\Gamma) \tag{3.5}$$

defines a differential on $\mathcal{G}$ and $\mathcal{G}_r$.

Definition 3.6. Kontsevich's graph complex is defined as $(\mathcal{G}, d)$, and his ribbon graph complex is defined as $(\mathcal{G}_r, d)$.

Remark 3.7. We could alternatively have used the uncompleted graph complex above; however, the completed version is the one which naturally contains characteristic classes of L_∞- or A_∞-algebras. In particular, taking homology commutes with taking completion, for the following well-known reason: We can write the uncompleted graph complex as a direct sum of the subcomplexes of graphs of a fixed genus (i.e., first Betti number of the graph as a topological space). For each fixed genus, the completion with respect to number of edges is the same as the completion with respect to the grading, i.e., the number of vertices, since there are only finitely many graphs with a fixed genus and number of vertices. Hence,

the completed graph complex is the same as the direct product of the completions of these subcomplexes with respect to their usual grading.

Finally, given a cyclic A_∞-algebra V, one constructs an element of $\mathcal{G}_r$, given by a sum

$$[V] := \sum_{\Gamma} \frac{1}{|\operatorname{Aut}(\Gamma)|} c_\Gamma(V) \cdot \Gamma, \tag{3.8}$$

where we sum over isomorphism classes of ribbon graphs Γ (with group of automorphisms $\operatorname{Aut}(\Gamma)$), and c_Γ is given as follows. Equip Γ with an orientation. We will define c_Γ so that the opposite orientation would produce $-c_\Gamma$. Namely, c_Γ is given by contracting the multiplications m_i of V with the pairings $(-,-)$ according to the graph. In more detail, consider

$$\prod_{i=1}^{n} (m(-),-) : \bigotimes_{i=1}^{n} V^{\otimes |\phi_V^{-1}(v_i)|} \to \mathbb{C}. \tag{3.9}$$

Let $h_1, \ldots, h_{|H|}$ be the ordering of the half-edges and $v_1, \ldots, v_{|V|}$ the ordering of the vertices defined by the orientation, and assume that the sign is 1. Let us pick *ciliations* of each of the vertices $v_1, \ldots, v_{|V|}$, which means a linear ordering of the half-edges $\varphi_V^{-1}(v_i)$ meeting each vertex v_i, compatible with the cyclic ordering given by the ribbon structure. Up to changing the sign, let us assume that the ordering of the half-edges is $\varphi_V^{-1}(v_1), \varphi_V^{-1}(v_2), \ldots, \varphi_V^{-1}(v_{|V|})$. Let $f \in V \otimes V$ be the inverse to the pairing $(-,-) : V \otimes V \to \mathbb{C}$. Finally, pick an arbitrary ordering of the edges $e_1, \ldots, e_{|E|}$. Then, one applies (3.9) to the element obtained by applying the signed permutation of components of $f^{\otimes |E|} \in V^{\otimes |H|}$ which rearranges the half-edges $\varphi_E^{-1}(e_1), \ldots, \varphi_E^{-1}(e_{|E|})$ into the ordering $h_1, \ldots, h_{|H|}$. One can check that the result does not depend on the choices of orderings (but only depends on the orientation of Γ by a sign, as mentioned above), and we let $c_\Gamma(V)$ to be the result of this computation.

In a similar manner, one constructs from any cyclic L_∞-algebra V an element $[V]$ of $\mathcal{G}$. Then, the following result is due to Kontsevich:

Proposition 3.10. [Kon94] If V is a cyclic A_∞- or L_∞-algebra then $[V]$ is a cycle on $(\mathcal{G}, d)$ or $(\mathcal{G}_r, d)$ respectively.

A direct proof of the proposition in the A_∞-case is contained, e.g., in [Igu04]. More conceptually, one can view a cyclic A_∞-algebra V as an algebra over $\mathsf{F}\underline{\mathscr{A}ss}^1$, the Feynman transform of the Det-twisted naive modular closure of the cyclic operad $\mathscr{A}ss$; see [GK98] and [CL07] concerning these notions. Therefore, we obtain a map of modular operads $\mathsf{F}\underline{\mathscr{A}ss}^1 \to \mathscr{E}(V)$, where $\mathscr{E}(V)$ is the modular endomorphism operad of V. The map between the vacuum parts of the corresponding operads

$$\mathsf{F}\underline{\mathscr{A}ss}^1((0)) \to \mathscr{E}(V)((0)) \cong \mathbb{C}$$

is precisely the characteristic class described above, and it follows that it does indeed give a cycle. One can prove the proposition similarly in the L_∞ case.

3.2. **Curved characteristic classes.** The preceding results have a natural generalization to the case of curved algebras. We need to remove the valence ≥ 2 condition, however, and study the graph complexes $(\widetilde{\mathcal{G}}, d), (\widetilde{\mathcal{G}}_r, d)$ of formal linear combinations of graphs and ribbon graphs where vertices are allowed to have valence 1 (we will not allow valence-zero vertices, since they don't add anything of value). We can consider this to be the complex of

"graphs with leaves," where a leaf is an edge incident to a valence-one vertex. Note that the conventional graph complex $(\mathcal{G}, d)$ is a subcomplex of $(\widetilde{\mathcal{G}}, d)$ and similarly in the ribbon case. Then, everything else goes through exactly as above, and we obtain the following result.

Proposition 3.11. Any curved cyclic A_∞- or L_∞-algebra V induces a cycle $[V]$ on $(\widetilde{\mathcal{G}}_r, d)$ or on $(\widetilde{\mathcal{G}}, d)$ respectively.

The curved graph homology could be expressed in terms of certain Gelfand-Fuks type homology. Namely, let W be a graded symplectic vector space and consider the graded formal Lie algebra $\mathfrak{g}(W)$ of formal symplectic vector fields on W. Similarly consider the graded formal Lie algebra $\mathfrak{g}_r(W)$ consisting of formal *noncommutative* symplectic vector fields on W, i.e. the Lie algebra $\mathrm{CDer}(\Pi W)$. Taking the stable limit as the dimension of the even or odd part of W goes to infinity, we arrive at following result (which is a straightforward adaptation of [Kon93, Theorem 1.1], up to technical problems stemming from the lack of complete reducibility of finite-dimensional representations of simple Lie superalgebras). Let $\mathbb{C}^m$ denote the even space of dimension m and $\Pi\mathbb{C}^m$ the odd space of dimension m. We equip $\mathbb{C}^{2m}$ with the standard symplectic form, and $\Pi\mathbb{C}^m$ with the standard odd symplectic (i.e., orthogonal) form.

Theorem 3.12. Let W be a fixed inner product space. There are isomorphisms

$$\mathrm{H}_\bullet(\widetilde{\mathcal{G}}_r) \cong \lim_{m\to\infty} \mathrm{HCE}_\bullet(\mathfrak{g}_r(W \oplus \mathbb{C}^{2m})), \quad \mathrm{H}_\bullet(\widetilde{\mathcal{G}}_r) \cong \lim_{m\to\infty} \mathrm{HCE}_\bullet(\mathfrak{g}_r(W \oplus \Pi\mathbb{C}^m)) \tag{3.13}$$

between the stable Chevalley-Eilenberg homology of the Lie algebra $\mathfrak{g}_r$ and of the corresponding version of the curved graph complex. Similarly, we have isomorphisms

$$H_\bullet(\widetilde{\mathcal{G}}) \cong \lim_{m\to\infty} \mathrm{HCE}_\bullet(\mathfrak{g}(W \oplus \mathbb{C}^{2m})), \quad H_\bullet(\widetilde{\mathcal{G}}) \cong \lim_{m\to\infty} \mathrm{HCE}_\bullet(\mathfrak{g}(W \oplus \Pi\mathbb{C}^m)). \tag{3.14}$$

Remark 3.15. It might be possible to further generalize this result to (certain) cases where both the even and the odd part have dimension going to infinity, but that creates additional technical difficulties that we prefer to avoid (as we do not need such generality). In any case, they are the same difficulties that arise in the original uncurved setting of [Kon93] (note that in *op. cit.* only the even case is considered).

Proof. The corresponding result for the uncurved graph complex and vector fields vanishing at the origin was established by Kontsevich [Kon93] and his proof carries over to the present context, up to some technical difficulties created by the fact that we are in the super context, where W_0 and W_1 can both be nonzero. Define a nonnegative grading on $\mathrm{CE}^\bullet(\mathfrak{g}_r(W))$ called *weight*, which is the sum of the homological grading and the degree of polynomial coefficients of the vector fields (this will correspond, on graphs, to the number of half-edges).

The main tool is that the inclusion of subcomplexes

$$\mathrm{CE}^\bullet(\mathfrak{g}_r(W \oplus \mathbb{C}^{2m}))^{\mathfrak{osp}(W\oplus\mathbb{C}^{2m})} \hookrightarrow \mathrm{CE}^\bullet(\mathfrak{g}_r(W \oplus \mathbb{C}^{2m})) \tag{3.16}$$

is asymptotically a quasi-isomorphism, and similarly replacing $\mathbb{C}^{2m}$ with $\Pi\mathbb{C}^m$. By this, we mean that in weights $\leq N$, there exists M such that, if $m \geq M$, then the inclusion is an isomorphism on homology in weights $\leq N$.

We carry out the argument with $\mathbb{C}^{2m}$; replacing this by $\Pi\mathbb{C}^m$ will not affect anything. Let $U_m := W \oplus \mathbb{C}^{2m}$. First note that $\mathfrak{osp}(U_m) \subseteq \mathfrak{g}_r(U_m)$ is the subspace of linear vector fields and hence acts trivially on the cohomology of $\mathrm{CE}^\bullet(\mathfrak{g}_r(U_m))$. Hence, the statement would

follow if it were true that $\mathfrak{osp}(U_m)$ acted completely reducibly on $\mathrm{CE}^\bullet(\mathfrak{g}_r(U_m))$. This is not, in general, true (when W is not purely even); however, it follows from Lemma 3.17 below that, for each $N \geq 0$, there exists M so that $m \geq M$ implies that the weight $\leq N$ part of $\mathrm{CE}^\bullet(\mathfrak{g}_r(U_m))$ is completely reducible as an $\mathfrak{osp}(U_m)$-representation. This is sufficient to deduce that (3.16) is a quasi-isomorphism in weights $\leq N$.

To complete the proof, it remains to relate the invariant subcomplex to graphs. This part of the argument is nearly identical to *op. cit.*, so we will be brief. Associated to each graph is an $\mathfrak{osp}(W)$-invariant element of $\mathrm{CE}_\bullet(\mathfrak{g}_r(W))$, as described in the previous subsection. However, the resulting map $\mathcal{G}_r \to \mathrm{CE}_\bullet(\mathfrak{g}_r(W))$ does not linearly extend to a map of complexes. Instead, if we attach to the dual of a graph in $\mathcal{G}_r^*$ a canonical element of $\mathrm{CE}^\bullet(\mathfrak{g}_r(W))^{\mathfrak{osp}(W)}$, as explained in *op. cit.*, one obtains a canonical map of complexes $(\mathcal{G}_r^*, d^*) \to \mathrm{CE}^\bullet(\mathfrak{g}_r(W))^{\mathfrak{osp}(W)}$. In more detail, under the identification $\mathcal{G}_r^* \cong \mathcal{G}_r$ using the basis of ribbon graphs (with fixed orientations), this map sends each oriented ribbon graph Γ to the corresponding functional which contracts elements of $\mathrm{CE}_{|V_\Gamma|}(\mathfrak{g}_r(W))$ using the symplectic form $W \otimes W \to \mathbb{C}$, similarly to the construction of §3.1. That is, we view $\mathrm{CE}_{|V_\Gamma|}(\mathfrak{g}_r(W)) \subset \mathrm{Hom}(W, \hat{T}(W))$ as a subspace of $\hat{T}(W)$ using the symplectic form, and contract with a permutation of $\omega^{\otimes |E_\Gamma|}$, where $\omega \in W \otimes W$ is the inverse to the symplectic form, so that the copy of ω corresponding to each edge contracts the corresponding pair of half-edges. For details on how to prove this indeed yields a map of complexes, see, e.g., [HL08, Theorem 4.10], and also [Ham06].

By the fundamental theorems of invariant theory [Wey39] (see also, e.g., [How95]) and the super generalization found in [Ser01], following the reasoning in the proof of Lemma 3.17 below, it follows that this map of complexes is asymptotically an isomorphism: for every fixed $N \geq 0$, there exists $M \geq 0$ so that, when $\dim W \geq M$, this map is an isomorphism if we restrict to graphs with at most N edges, and hence elements of $\mathrm{CE}^\bullet(\mathfrak{g}_r(W))^{\mathfrak{osp}(W)}$ of weight $\leq N$.

This construction goes through completely analogously for commutative graphs and the Lie algebra $\mathfrak{g}$. $\square$

As in the proof above, let $U_m := W \oplus \mathbb{C}^{2m}$. Also define $U_m' := W \oplus \Pi\mathbb{C}^m$.

Lemma 3.17. For all $N \geq 0$, there exists $M \geq 0$ such that, when $m \geq M$, $U_m^{\otimes N}$ and $(U_m')^{\otimes N}$ are completely reducible $\mathfrak{osp}(U_m)$- and $\mathfrak{osp}(U_m')$-modules, respectively.

Proof. We carry out the argument for U_m; the same applies for U_m'. Suppose that one has subrepresentations $0 \neq \tau_1 \subsetneq \tau_2 \subseteq U_m^{\otimes N}$. We wish to show that the inclusion $\tau_1 \hookrightarrow \tau_2$ splits. By adjunction, we equivalently need to show that the composition $\mathbb{C} \hookrightarrow \tau_1^* \otimes \tau_1 \hookrightarrow \tau_1^* \otimes \tau_2$ splits. Since $U_m^{\otimes N}$ is self-dual using the pairing, $\tau_1^* \otimes \tau_2 \subseteq U_m^{\otimes 2N}$. This means that it suffices to show that the inclusion of the invariant part, $(U_m^{\otimes 2N})^{\mathfrak{osp}(U_m)} \hookrightarrow U_m^{\otimes 2N}$, splits. Let us replace $2N$ by N for convenience.

This last fact follows using the fundamental theorems of invariant theory. Namely, by [Ser01], all of the $\mathfrak{osp}(U_m)$-invariants in the tensor algebra $T(U_m)$ are tensor products of the pairing on U_m with an invariant related to the determinant, whose tensor weight (i.e., number of tensor components) is a function of m that goes to infinity when m does. Hence, for large enough m, all of the invariants in weight $U_m^{\otimes \leq N}$ are spanned by tensor products of the pairing on W. By the classical second fundamental theorem of invariant theory [Wey39] (see also [How95]), for large enough m, there are no nontrivial relations between these invariants. The same argument applied to U_m^* shows that the coinvariants have the same description.

Moreover, for large enough m, the invariants of $U_m^{\otimes \le N}$ and invariants of $(U_m^*)^{\otimes \le N}$ have a perfect pairing. This implies that the composition $(U_m^{\otimes \le N})^{\mathfrak{osp}(U_m)} \hookrightarrow U_m^{\otimes N} \twoheadrightarrow (U_m^{\otimes N})_{\mathfrak{osp}(U_m)}$ is an isomorphism for large enough m. $\square$

Unlike the case for the usual graph homology, it is possible to give a complete calculation of the homology of the curved graph complex and curved ribbon graph complex:

Theorem 3.18.

(a) The homology of the complex $(\widetilde{\mathcal{G}}_r, d)$ is identified with the space of formal linear combinations of graphs all of whose connected components are graphs whose vertices have valence one with the exception of at most a single vertex, which has odd valence.

(b) The homology of the complex $(\widetilde{\mathcal{G}}, d)$ is identified with the space of formal linear combinations of graphs each of whose connected components is the connected graph with one edge and two vertices.

We can reinterpret this theorem as follows. First, it is enough to compute the homology of the subcomplex of connected nonempty graphs, $\widetilde{\mathcal{G}_{r,c}} \subset \widetilde{\mathcal{G}}_r$ and $\widetilde{\mathcal{G}}_c \subset \widetilde{\mathcal{G}}$. We call these complexes the *connected* ribbon graph complex and the connected graph complex. Note that $\widetilde{\mathcal{G}}_r \cong \operatorname{Sym} \widetilde{\mathcal{G}_{r,c}}$ and $\widetilde{\mathcal{G}} \cong \operatorname{Sym} \widetilde{\mathcal{G}}_c$. Then, the desired result is that the homology of the former is identified with linear combinations of star-shaped graphs with an odd number of edges and at most a single vertex of valence ≥ 2, and the latter is one-dimensional and spanned by the connected graph with two vertices and a single edge.

In other words, the theorem states that $\widetilde{\mathcal{G}_{r,c}}$ and $\widetilde{\mathcal{G}}_c$ are quasi-isomorphic to the subcomplexes spanned by connected graphs with at most a single vertex of valence ≥ 2. These complexes are identical with the deformation complexes of Theorem 2.8 for the one-dimensional curved A_∞ and L_∞ algebras with curvature c satisfying $(c, c) = 1$, and all higher operations zero. That is, they are the graded vector spaces of noncommutative symplectic vector fields and ordinary symplectic vector fields on the odd one-dimensional symplectic vector space $\mathbb{C} \cdot c$, equipped with the differential $\operatorname{ad}(\frac{\partial}{\partial c})$ (which turns out to be zero).

Remark 3.19. The stable homology of the Lie algebra of symplectic vector fields has been computed by Guillemin and Shnider in [GS73], and thus, part (b) of the above theorem could be deduced from their calculation, taking into account Theorem 3.12. However we include this result for completeness, and because the argument we use in part (a) essentially extends without change to this case.

Proof of Theorem 3.18. Call a graph a *line segment* if it is topologically a line segment, i.e., it is connected, and either it is a single vertex with no edges, or all of its vertices have valence two except for two vertices, which have valence one. Given a connected graph Γ, let us call a vertex $v \in V_\Gamma$ *exterior* if it either has valence at most one, or one of the connected components of $\Gamma \setminus v$ is a line segment. Call all other vertices *interior*.

We will make use of a filtration on the connected (ribbon) graph complex given by the number of interior vertices in the graph. We call this the *interior vertex filtration*. The associated graded complex is identified, as a vector space, with the (ribbon) graph complex, and with the differential which is almost the same, but only contracts edges incident to at least one exterior vertex. Moreover, the associated graded complexes $\mathsf{gr}\,\widetilde{\mathcal{G}_{r,c}}$ and $\mathsf{gr}\,\widetilde{\mathcal{G}}_c$ are graded not only by number of interior vertices, but by the subgraph $\Gamma_0 \subseteq \Gamma$ obtained by restricting to interior vertices and edges which are incident only to interior vertices (allowing

here also the empty graph and the graph with a single vertex and no edges). It is clear that Γ_0 is connected (possibly empty). Let $\mathsf{gr}_{\Gamma_0} \widetilde{\mathcal{G}_{r,c}}$ or $\mathsf{gr}_{\Gamma_0} \widetilde{\mathcal{G}}_c$ denote the resulting subcomplex graded by Γ_0.

(a) We claim that, when Γ_0 contains at least one edge, then $\mathsf{gr}_{\Gamma_0} \widetilde{\mathcal{G}_{r,c}}$ is acyclic.

Assume that Γ_0 contains an edge. Pick a half-edge h of Γ_0, and let v be the incident vertex. We construct from v and h a contracting homotopy s on $\mathsf{gr}_{\Gamma_0} \widetilde{\mathcal{G}_{r,c}}$. Namely, for every oriented graph Γ whose subgraph on internal vertices is Γ_0, let $s\Gamma$ be the graph obtained from Γ by adding a new univalent vertex together with an edge connecting it to v. The resulting new half-edge incident to v is, in the cyclic ordering at v, one half-edge counterclockwise away from h. Pick the orientation on the new graph $s\Gamma$ so that Γ is one of the summands of $d(s\Gamma)$.

We claim that $sd + ds = \mathrm{Id}$. It suffices to show that, for every oriented graph Γ as above, $sd(\Gamma) + ds(\Gamma) = \Gamma$. In turn, it suffices to show that, for every edge $e \in E_\Gamma$, $sd_e\Gamma = -d_e(s\Gamma)$. This follows immediately from our definition of s.

Hence, we deduce that $\mathsf{gr}_{\Gamma_0} \widetilde{\mathcal{G}_{r,c}}$ is acyclic, as claimed.

Next, we compute $\mathsf{gr}_{\Gamma_0} \widetilde{\mathcal{G}_{r,c}}$ where Γ_0 is either empty or is the graph with a single vertex and no edges. Call the first graph "$\emptyset$" and the second one "pt."

First, $\mathsf{gr}_{\mathrm{pt}} \widetilde{\mathcal{G}_{r,c}}$ consists of star-shaped graphs with a single vertex of valence ≥ 3, with the usual graph differential except that we do not allow to contract an edge that would result in a graph without a vertex of valence ≥ 3, i.e., an edge which is incident to a vertex of valence 1 and a vertex of valence 3. Consider the filtration by the valence of the interior vertex. The associated graded complexes have homology which is one-dimensional, concentrated in the part where all vertices but one have valence 1. Moreover, such graphs are actually zero when the node has even valence (because a cyclic symmetry can reverse the orientation). Hence, the associated spectral sequence computing $H_*(\mathsf{gr}_{\mathrm{pt}} \widetilde{\mathcal{G}_{r,c}})$ collapses at the first page, and the resulting homology is spanned by the star-shaped graphs with a single vertex of odd valence ≥ 3 and with all other vertices of valence 1.

Next, $\mathsf{gr}_{\emptyset} \widetilde{\mathcal{G}_{r,c}}$ is the subcomplex of line segments, whose homology is one-dimensional and spanned by the line segment with two vertices.

We deduce that the first page of the spectral sequence of the interior vertex filtration on $\widetilde{\mathcal{G}_{r,c}}$ is concentrated in the part with ≤ 1 interior vertices. The part in degree 1 is the span of star-shaped graphs with one vertex of odd valence ≥ 3 and the other vertices of valence 1, and the part in degree 0 is the span of the line segment with two vertices. Since all of these graphs have only odd numbers of edges, it is clear that the spectral sequence collapses at the first page, and the graphs above span the homology of $\widetilde{\mathcal{G}_{r,c}}$, as desired.

(b) The same argument as above applies in this case, except that, since our graphs no longer have cyclic orderings of half edges at vertices, we modify the construction of the contracting homotopy s (used to show that $\mathsf{gr}_{\Gamma_0} \widetilde{\mathcal{G}_{r,c}}$ is acyclic when Γ_0 contains an edge) accordingly. Namely, we remove the condition that the new edge be next to h in the counterclockwise cyclic ordering of half edges at the vertex v. Everything else goes through without change, except that now the star-shaped graphs with a single vertex of valence ≥ 1 are all zero except for the one with only two vertices. This implies the desired result. $\square$

We see that the complex of simply-connected graphs (which splits off as a direct summand) carries all of the homology of the complexes $(\widetilde{\mathcal{G}}_r, d)$ and $(\widetilde{\mathcal{G}}, d)$, and we obtain the following result.

Corollary 3.20. The inclusions of complexes $(\mathcal{G}_r, d) \subset (\widetilde{\mathcal{G}}_r, d)$ and $(\mathcal{G}, d) \subset (\widetilde{\mathcal{G}}, d)$ induce the zero maps on homology.

Remark 3.21. It is natural to ask whether the nontrivial homology classes in the curved graph complexes $\widetilde{\mathcal{G}}_r$ and $\widetilde{\mathcal{G}}$ are detected by curved A_∞- and L_∞-algebras. The answer is yes; in fact, they are detected by one-dimensional algebras. For the A_∞ case, let $\Gamma(i)$ be the star-shaped graph whose central vertex has valence $2i+1$ and $V(i)$ be the one-dimensional cyclic curved A_∞-algebra spanned by an even vector c such that $(c,c) = 1$, $m_0 = c$, $m_{2i}(c, \ldots, c) = c$, and all other A_∞-products are zero. Then it is easy to see that the cycle $[V(i)] \in \widetilde{\mathcal{G}}_r$ is homologous to $\pm\frac{1}{2i+1}\Gamma(i)$ (the sign depends on the choice of an orientation on $\Gamma(i)$). Next, let $\Gamma(0)$ be the connected graph with one edge and two vertices. Then, we may let $V(0)$ be the one-dimensional cyclic curved A_∞-algebra spanned by an even vector c such that $m_0 = c$, $(c,c) = 1$, and all the higher A_∞-products are zero. Then, $[V(0)] \in \widetilde{\mathcal{G}}_r$ is again homologous to $\pm\frac{1}{2}\Gamma(0)$. The same statement holds for the L_∞-setting, if we now let $\Gamma(0)$ be an ordinary (not ribbon) connected graph also with two vertices and one edge, and consider the one-dimensional curved cyclic L_∞-algebra, $V(0)$, with curvature c satisfying $(c,c) = 1$ and all higher operations equal to zero: $[V(0)] \in \widetilde{\mathcal{G}}_r$ is homologous to $\pm\frac{1}{2}\Gamma(0)$.

4. Homology of Lie algebras of vector fields and stability maps

In order to refine Theorem 3.18, we recall first a more general way to view the construction of characteristic classes. Let $\mathfrak{g}$ be a formal dgla. Consider a Maurer-Cartan element of $\mathfrak{g}$, i.e., an element $x \in \Pi\mathfrak{g}$ satisfying $dx + \frac{1}{2}[x,x] = 0$. Then, $e^x = \sum_{i\geq 0} x^i/i!$ defines a cycle in $\mathrm{CE}_\bullet(\mathfrak{g})$, and hence a homology class of even total degree. In the situation where the differential d on $\mathfrak{g}$ is zero, using the homological degree $|S^i\Pi\mathfrak{g}| = i$, each element x^i itself is a cycle, and we obtain *unstable* characteristic classes $[x^i] \in \mathrm{CE}_i(\mathfrak{g})$.

Moreover, if $\mathfrak{h} \subseteq \mathfrak{g}_0$ is a pronilpotent Lie subalgebra of the even part of $\mathfrak{g}$, then there is defined a notion of *gauge equivalence* of Maurer-Cartan elements corresponding to the adjoint action of the Lie group of the Lie algebra $\mathfrak{h}$; then it follows that if two Maurer-Cartan elements are gauge equivalent by a gauge in $\mathfrak{h}$, then their characteristic classes are homologous. (One can more generally take $\mathfrak{h} \subseteq \mathfrak{g}_0$ to be a Lie subalgebra which is the Lie algebra of a pro-Lie group). This statement, as well as a more detailed treatment of characteristic classes, can be found in, e.g., [Ham09].

Returning to the situation of a (cyclic or curved) L_∞- or A_∞-algebra V, the corresponding element of the Lie algebra $\mathrm{Der}^0(\hat{S}(\Pi V^*)), \mathrm{Der}(\hat{T}(\Pi V^*))$, etc., defines a canonical homology class.

The relationship to the aforementioned characteristic classes is Kontsevich's result that the limit as the dimension of V goes to infinity of the Lie homology of $\mathrm{CDer}^0(\hat{S}(\Pi V^*))$ or $\mathrm{CDer}^0(\hat{T}(\Pi V^*))$ is the completion (by number of edges) of the homology of the graph complexes $(\mathcal{G}, d)$. In the curved situation the relevant result is Theorem 3.12.

Furthermore, given V, and any inner product space W, we can form the trivial extension $V \oplus W$ where all multiplications with the second factor are zero. This induces maps

$$\varphi_{V,W} : \mathrm{CE}_\bullet(\mathrm{CDer}(\hat{S}(\Pi V^*))) \to \mathrm{CE}_\bullet(\mathrm{CDer}(\hat{S}(\Pi(V \oplus W)^*))), \tag{4.1}$$

$$\varphi_{V,W} : \mathrm{CE}_\bullet(\mathrm{CDer}(\hat{T}(\Pi V^*))) \to \mathrm{CE}_\bullet(\mathrm{CDer}(\hat{T}(\Pi(V \oplus W)^*))), \tag{4.2}$$

and similarly the restrictions $\varphi^0_{V,W} := \varphi_{V,W}|_{\mathrm{CE}_\bullet(\mathrm{CDer}^0(\hat{S}(\Pi V^*)))}$ or $\varphi^0_{V,W} := \varphi_{V,W}|_{\mathrm{CE}_\bullet(\mathrm{CDer}^0(\hat{T}(\Pi V^*)))}$. It is well known (and easy to check) that Kontsevich's construction (§3.1) is obtained from the above construction in the limit: the image of the unstable characteristic cycle $\sum_i \xi^i/i! \in \mathrm{CE}_\bullet(\mathrm{CDer}(\hat{S}(\Pi V^*)))$ under $\varphi^0_{V,W}$ as $\dim W_0 \to \infty$ (for fixed W_1) identifies with the characteristic cycle on $(\mathcal{G}, d)$ given in §3.1; similarly if we fix W_0 and let $\dim W_1 \to \infty$. In the curved setting, by Theorem 3.18, for each fixed degree i, if we fix W_1, then for large enough $\dim W_0$, $\varphi_{V,W}$ induces a projection on homology,

$$\mathrm{HCE}_i(\mathrm{CDer}(\hat{S}(\Pi V^*))) \twoheadrightarrow \begin{cases} \mathbb{C}, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

The same is true if we fix W_0 and consider large enough $\dim W_1$. A similar result is true in the associative version, where now we project onto the span of graphs whose connected components are stars with odd valence as in Theorem 3.18.

Using Theorem 2.8, we can prove an unstable analogue of Corollary 3.20, which gives information about the maps $\varphi_{V,W}$ for all W with $W_0 \neq 0$:

Theorem 4.3. If V and W are inner product spaces and $W_0 \neq 0$, then the compositions

$$\mathrm{CE}_i(\mathrm{CDer}^0(\hat{S}(\Pi V^*))) \to \mathrm{CE}_i(\mathrm{CDer}(\hat{S}(\Pi V^*))) \overset{\varphi_{V,W}}{\to} \mathrm{CE}_i(\mathrm{CDer}(\hat{S}(\Pi(V \oplus W)^*))), \tag{4.4}$$

$$\mathrm{CE}_i(\mathrm{CDer}^0(\hat{T}(\Pi V^*))) \to \mathrm{CE}_i(\mathrm{CDer}(\hat{T}(\Pi V^*))) \overset{\varphi_{V,W}}{\to} \mathrm{CE}_i(\mathrm{CDer}(\hat{T}(\Pi(V \oplus W)^*))) \tag{4.5}$$

are zero on homology.

Before we prove the theorem, we first prove a lemma which may be interesting in itself. Given an element $c \in V$ and a L_∞- or A_∞-structure given by a Maurer-Cartan element $\xi \in \mathrm{CDer}^0(\hat{S}(\Pi V^*))$ or $\xi \in \mathrm{CDer}^0(\hat{T}(\Pi V^*))$, we say that c is *central* if $[c, \xi] = 0$. In particular, for an ordinary Lie algebra, c is an element satisfying $\{c, v\} = 0$ for all $v \in V$, and for an ordinary associative algebra, c satisfies $c \cdot v = v \cdot c$ for all $v \in V$.

Lemma 4.6. Suppose that V is an (uncurved) L_∞- (or A_∞-) algebra with a nonzero even central element $c \in V_0$. Then, the image of the resulting (unstable) characteristic class of V under the appropriate map,

$$\mathrm{CE}_\bullet(\mathrm{CDer}^0(\hat{S}(\Pi V^*))) \to \mathrm{CE}_\bullet(\mathrm{CDer}(\hat{S}(\Pi V^*))) \text{ or} \tag{4.7}$$

$$\mathrm{CE}_\bullet(\mathrm{CDer}^0(\hat{T}(\Pi V^*))) \to \mathrm{CE}_\bullet(\mathrm{CDer}(\hat{T}(\Pi V^*))), \tag{4.8}$$

is zero on homology.

Proof. We first consider the L_∞ case. Let $\mathfrak{g} := \mathrm{CDer}(\hat{S}(\Pi V^*))$. Let $\xi \in \mathrm{CDer}^0(\hat{S}(\Pi V^*)) \subseteq \mathfrak{g}$ correspond to the L_∞-structure on V, i.e., ξ satisfies $[\xi, \xi] = 0$, and viewed as an element of $\mathfrak{g}$, $\xi(v)$ has no constant term for all $v \in \Pi V^*$. Let ℓ denote the structure maps for the algebra V, with $\ell_i : V^{\otimes i} \to V$ the i-th component.

Now, consider the L_∞-structure $\{\ell_i^\lambda\}$ on V which is obtained by $\ell_i^\lambda := \ell_i$ if $i \geq 1$, and $\ell_0^\lambda = \lambda c$ for $\lambda \in \mathbb{C}$. Since c is central and even, these indeed define L_∞-structures.

Then, by Theorem 2.8, V equipped with ℓ^λ is gauge equivalent to an algebra with all higher multiplications equal to zero, and curvature equal to λc. This gauge equivalence is by vector fields with zero constant and linear term, which form a pronilpotent dgla. Hence, in the limit as $\lambda \to 0$, we see that the characteristic class of (V, ℓ^λ) becomes a boundary, i.e., the original characteristic class of (V, ℓ) is a boundary in $\mathrm{CE}_\bullet(\mathrm{CDer}(\hat{S}(\Pi V^*)))$, as desired.

In the A_∞ case, the same argument applies: as before, one deforms the uncurved A_∞-structure $\{m_i\}$ to the curved structure $\{m_i^\lambda\}$ with $m_i^\lambda = m_i$ for $i \geq 1$ and $m_0^\lambda = \lambda c$. The difference is, by Theorem 2.8, the resulting structure (V, m^λ) is gauge equivalent to the algebra described there, which does not have all higher operations zero. Call this algebra structure $\{(m')_i^\lambda\}$. Even though these are nonzero for infinitely many i, it is still true that, as $\lambda \to 0$, $(m')_i^\lambda \to 0$, and so we still deduce that the original characteristic class was a boundary. $\square$

Proof of Theorem 4.3. We treat only the L_∞ case, since the A_∞ case is identical. If V is an uncurved L_∞-algebra, then $V \oplus W$ is an L_∞-algebra with a nonzero central element, namely, any nonzero element of W_0. Hence, the image of the resulting characteristic class under (4.4) is zero.

However, in general, not all homology classes of $\mathrm{CE}_\bullet(\mathrm{CDer}^0(\hat{S}(\Pi V^*)))$ are obtainable in the above manner. To fix this problem, we can consider nontrivial coefficients. Given any cdga $\mathfrak{a}$, we may consider $\mathfrak{a}$-linear L_∞-algebra structures on $U := V \otimes_{\mathbb{C}} \mathfrak{a}$, for a fixed finite-dimensional space V. Denote $U^* := \mathrm{Hom}_{\mathfrak{a}}(U, \mathfrak{a})$. Then, these algebra structures on U are, by definition, Maurer-Cartan elements of $\mathrm{Der}^0(\hat{S}_{\mathfrak{a}}(\Pi U^*))$, i.e., odd elements f satisfying $df + \frac{1}{2}[f, f] = 0$, where d is the differential induced by the differential on $\mathfrak{a}$. If we fix an inner product on V, we obtain an induced $\mathfrak{a}$-linear inner product on U (of the form $U \otimes_{\mathfrak{a}} U \to \mathfrak{a}$), and cyclic L_∞-algebras of this form are given by elements of $\mathrm{CDer}^0(\hat{S}_{\mathfrak{a}}(\Pi U^*))$.

As above, we obtain an unstable characteristic class of $\mathrm{CE}_\bullet(\mathrm{CDer}^0(\hat{S}_{\mathfrak{a}}(\Pi U^*)))$. The above argument shows that the images of such classes under the tensor product of (4.4) with $\mathrm{Id}_{\mathfrak{a}}$ are boundaries.

To conclude the proof, we will use a universal example. Quite generally, if $\mathfrak{h}$ is a formal dgla and $\mathfrak{a}$ a cdga, then $\mathrm{CE}^\bullet(\mathfrak{h}) = S\Pi\mathfrak{h}^*$ is naturally a cdga, and Maurer-Cartan elements of $\mathfrak{h} \otimes \mathfrak{a}$ identify with cdga morphisms $\mathrm{CE}^\bullet(\mathfrak{h}) \to \mathfrak{a}$. If we set $\mathfrak{a} := \mathrm{CE}^\bullet(\mathfrak{h})$, then the identity map $\mathrm{Id}_{\mathrm{CE}^\bullet(\mathfrak{h})}$ yields a Maurer-Cartan element of $\mathfrak{h} \otimes \mathfrak{a}$, and the resulting cycle $\sum_i \xi^i / i! \in \widehat{\mathrm{CE}_\bullet(\mathfrak{h}) \otimes \mathfrak{a}} \cong \mathrm{End}(\mathrm{CE}^\bullet(\mathfrak{h}))$ is nothing but the identity map.

Applying this construction to the case $\mathfrak{h} = \mathrm{CDer}^0(\hat{S}(\Pi V^*))$ and $\mathfrak{a} = \mathrm{CE}^\bullet(\mathrm{CDer}^0(\hat{S}(\Pi V^*)))$, we deduce from the above that the $\mathfrak{a}$-linear maps $\widehat{\varphi_{V,W} \otimes \mathrm{Id}_{\mathfrak{a}}}$ send the cycle ξ corresponding to the identity element of $\mathrm{End}(\mathrm{CE}^\bullet(\mathfrak{h}))$ to a boundary. This implies that (4.4) itself is zero on homology, as desired. $\square$

5. Operadic generalization

In this section we sketch an operadic generalization of our main results, from the associative and Lie cases to more general settings. As we will show in the following section, these include Poisson, Gerstenhaber, BV, permutation, and pre-Lie algebras: see Examples 6.1, 6.2, and 6.3 in the following section.

We may think of curved (cyclic) A_∞- and L_∞-algebras as arising from the following construction, which we think of heuristically as a type of "Koszul duality" between operads governing curved algebras and those governing unital algebras (we do not attempt to make this description precise).

Remark 5.1. For a somewhat related result, see [HM10], where resolutions for operads of unital algebras are constructed by defining a Koszul dual curved cooperad and performing

a version of the cobar construction. Here, we will not make use of the notion of curved (co)operads defined in [HM10], and will only use ordinary (dg) operads.

Let $\mathcal{O}$ be a (cyclic) dg operad [GK95], which we assume to be unital with unit $I \in \mathcal{O}(1)$. Moreover, we will assume throughout that each $\mathcal{O}(i)$ is a finite-dimensional $\mathbb{Z}/2$-graded vector space (this isn't really essential, but it makes dualization less technical, and includes all operads we have in mind. Properly speaking, we view $\mathcal{O}(i)$ as a formal space, and everything generalizes to the infinite-dimensional formal setting.) Let $m_0 \in \mathcal{O}(0)$ be an element (corresponding to a "0-ary" operation). Recall that every (cyclic) operad is an $\mathbb{S}$ ($\mathbb{S}_+$)-module, where an $\mathbb{S}$-module is defined as a collection $\{V_m\}_{m\geq 0}$ of S_m-modules for all $m \geq 0$, and an $\mathbb{S}_+$-module is an $\mathbb{S}$-module where each V_m is actually a module over S_{m+1} (the underlying $\mathbb{S}$-module is obtained using the inclusion $S_m \subseteq S_{m+1}$ of permutations fixing $m+1$).

We will now consider $\mathcal{O}$ as a nonunital operad and perform the cobar construction [GK94]. Namely, let $\mathcal{O}^* := \{\mathcal{O}(m)^*\}$ be the dual $\mathbb{S}$ (or $\mathbb{S}_+$)-module. Let $C(\mathcal{O})$ be the free operad generated by $\Pi\mathcal{O}^*$, equipped with a differential $d_{C(\mathcal{O})}$ obtained as follows. For every $j \geq 0$ and all $1 \leq i \leq k$, there is an operadic composition map

$$\circ_i^{k,j} : \mathcal{O}(k) \otimes \mathcal{O}(j) \to \mathcal{O}(j+k-1), \tag{5.2}$$

which corresponds to plugging the element of $\mathcal{O}(j)$ into the i-th input of the element of $\mathcal{O}(k)$. Let $(\circ_i^{k,j})^*$ be the linear dual to the above map. Then, we define the differential $d_{C(\mathcal{O})}$ to be the unique extension to a derivation of the operation

$$d_{C(\mathcal{O})}|_{\mathcal{O}(i)^*} = d_{\mathcal{O}}^* + \bigoplus_{j,k} (\circ_i^{k,j})^*. \tag{5.3}$$

Now, in the case that $\mathcal{O}$ is the ordinary (non-dg) operad governing *unital* associative or commutative algebras, then $C(\mathcal{O})$ is a dg operad with the property that graded (dg) algebras V over $C(\mathcal{O})$, equipped with zero differential, are the same as curved A_∞- or L_∞-algebras. In this case, $m_0 \in \mathcal{O}(0)$ is the unit of the multiplication, and the curvature of a graded algebra V over $C(\mathcal{O})$ is the image of $m_0^* \in \mathcal{O}(0)^*$ in V.

Next, suppose that $\mathcal{O}'$ is the suboperad of $\mathcal{O}$ in positive arity, i.e., $\mathcal{O}'(0) = 0$ and $\mathcal{O}'(i) = \mathcal{O}(i)$ for $i \geq 1$. Further suppose that $\mathcal{O}'$ is a Koszul operad with augmentation $\overline{\mathcal{O}'}$ (i.e., $\overline{\mathcal{O}'}$ is a suboperad such that $\mathcal{O}'(1) = \overline{\mathcal{O}'}(1) \oplus \mathbb{C} \cdot \mathrm{Id}$, and $\overline{\mathcal{O}'}(i) = \mathcal{O}'(i)$ for $i \neq 1$). Then, $C(\overline{\mathcal{O}'})$ yields a resolution of the Koszul dual operad $(\mathcal{O}')^!$ (this last fact is equivalent to Koszulity; see [GK94]). *Caution:* This operad $C(\overline{\mathcal{O}'})$ is sometimes called the cobar construction on the unital operad $\mathcal{O}'$ itself, but here we consider cobar constructions only on nonunital operads.

We think of an algebra over $C(\mathcal{O})$ (with zero differential) as a certain type of *curved* version of an $((\mathcal{O}')^!)_\infty$-algebra, as opposed to an algebra over $C(\overline{\mathcal{O}'})$, which, in the case $\mathcal{O}'$ is Koszul, is the same as an ordinary $((\mathcal{O}')^!)_\infty$-algebra. (More generally, given a dg space V with possibly nontrivial differential, one could view a $C(\mathcal{O})$-algebra structure on it as a dg generalization of a curved $((\mathcal{O}')^!)_\infty$-algebra; thus one could speak, for instance, about a curved dg A_∞- or L_∞-algebra. This would be a Maurer-Cartan element in the appropriate differential graded Lie algebras of (formal, noncommutative) vector fields rather than simply square-zero odd derivations.)

Let us return to the general setting of an arbitrary (cyclic) operad $\mathcal{O}$. The analogues of Theorems 2.1 and 2.8 are then the following.

Definition 5.4. A *weakly unital multiplication* $m_2 \in \mathcal{O}(2)$ is an element such that the image of the maps

$$m_2(- \otimes \mathrm{Id}), m_2(\mathrm{Id} \otimes -) : \mathcal{O}(0) \to \mathcal{O}(1) \tag{5.5}$$

lie in the one-dimensional space $\mathbb{C} \cdot \mathrm{Id} \subseteq \mathcal{O}(1)$, and such that the maps are not both zero.

If m_2 is weakly unital, then

$$m_2(- \otimes \mathrm{Id}) = m_2(\mathrm{Id} \otimes -). \tag{5.6}$$

Indeed, if $m_0 \in \mathcal{O}(0)$ is any element such that $m_2(m_0 \otimes \mathrm{Id}) = \lambda \cdot \mathrm{Id}$ for some nonzero $\lambda \in \mathbb{C}$, then it follows that $m_2(m_0 \otimes m_0) = \lambda \cdot m_0$, and hence that $m_2(\mathrm{Id} \otimes m_0) = \lambda \cdot \mathrm{Id}$.

Therefore, when m_2 is weakly unital, we define $\widetilde{m_2} \in \mathcal{O}(0)^*$ to be the resulting map (5.6).

Next, given a $C(\mathcal{O})$ algebra V, the 0-ary operations form a map $\gamma : \mathcal{O}(0)^* \to V$, which we call the 0*-ary structure map*.

Definition 5.7. Given an operad $\mathcal{O}$ with weakly unital multiplication m_2, a $C(\mathcal{O})$ algebra V with 0-ary structure map γ is said to be *nontrivially curved with respect to* m_2 if the element $\gamma(\widetilde{m_2})$ is nonzero.

We call the element $\gamma(\widetilde{m_2})$ the *curvature* of V (for the element m_2).

Then, Theorems 2.1 and 2.8 generalize as follows. Fix a graded vector space V (with zero differential). If $\mathcal{O}$ is any dg operad, we denote a $\mathcal{O}$-algebra structure on V as a pair (V, ϕ) for the algebra structure, where given any $o \in \mathcal{O}(k)$, $\phi(o) \in \mathrm{Hom}_{\mathbb{C}}(V^{\otimes k}, V)$.

Claim 5.8. Fix a graded vector space V.

(i) Let $\mathcal{O}$ be an operad with a weakly unital multiplication m_2. Let (V, ϕ) be a $C(\mathcal{O})$-algebra structure with nonzero curvature $c \in V$. Then, (V, ϕ) is gauge equivalent to the algebra (V, ϕ') with the same 0-ary structure map γ, but with all higher multiplications equal to zero.

(ii) Let $\mathcal{O}$ be a cyclic operad with a weakly unital multiplication m_2. Let (V, ϕ) be a cyclic $C(\mathcal{O})$-algebra structure, with nonzero curvature $c \in V$. Let $c' \in V$ be an element such that $(c', c) = 1$. Then, (V, ϕ) is gauge equivalent to the algebra (V, ϕ') with the same 0-ary structure map γ, but with higher multiplications of the form

$$\phi'(m)(v_1, \ldots, v_i) = \Big(\prod_{j=1}^{i} (c', v_j)\Big) \cdot (\phi(m)(c, c, \ldots, c), c) \cdot c'. \tag{5.9}$$

Here, gauge equivalence is defined as follows. For any $C(\mathcal{O})$-algebra structure on V, one can form the formal dg $\mathcal{O}$-algebra $\hat{C}_{\mathcal{O}}(V)$, which is defined as the completed free $\mathcal{O}$-algebra generated by ΠV^*:

$$\hat{C}_{\mathcal{O}}(V) = \prod_{n=0}^{\infty} \mathcal{O}(n) \otimes_{S_n} (\Pi V^*)^{\otimes n},$$

equipped with the differential $d_{\hat{C}_{\mathcal{O}}(V)}$ on $\hat{C}_{\mathcal{O}}(V)$ defined by the canonical linear map $\Pi V^* \to \hat{C}_{\mathcal{O}}(V)$, which is shifted dual to a restriction of the structure maps $C(\mathcal{O}) \to \mathrm{Hom}(V^{\otimes n}, V)$.

Next, assume for the moment that V is equipped with the zero $C(\mathcal{O})$-algebra structure, i.e., $d_{\hat{C}_{\mathcal{O}}(V)} = 0$. We may then form the formal dgla $\mathrm{Der}(\hat{C}_{\mathcal{O}}(V))$, where the differential is induced by the differential on V (and is zero in the case where V was merely a graded vector

space, as in the setting of (non-dg) curved A_∞- or L_∞-algebras considered in earlier sections of this paper). It is then a standard fact (see, e.g., [GJ94, Proposition 2.15], Proposition 2.15 where, however, this result is formulated in the language of coalgebras) that the above yields a bijection between square-zero odd derivations of $\hat{C}_{\mathcal{O}}(V)$ and $C(\mathcal{O})$-structures on V. Finally, we define two $C(\mathcal{O})$-structures on V to be gauge equivalent if the corresponding differentials $d_{\hat{C}_{\mathcal{O}}(V)}$ and $d'_{\hat{C}_{\mathcal{O}}(V)}$ are gauge equivalent, i.e., that there exists an even derivation $\xi \in \mathrm{Der}^0(\hat{C}_{\mathcal{O}}(V))$ with zero constant term such that $d'_{\hat{C}_{\mathcal{O}}(V)} = e^{\mathrm{ad}\,\xi} d_{\hat{C}_{\mathcal{O}}(V)}$.

Similarly, if $\mathcal{O}$ is a cyclic operad, and V is a cyclic algebra over $\mathcal{O}$ (i.e., an algebra equipped with a nondegenerate inner product compatible with the cyclic structure on $\mathcal{O}$), then there is a natural $\mathbb{Z}/(m+1)$ action on the subspace $\Pi V^* \otimes \mathcal{O}(m) \otimes_{S_m} (\Pi V^*)^{\otimes m} \cong \mathrm{Hom}(\Pi V^*, \hat{C}_{\mathcal{O}}(V)) = \mathrm{Der}(\hat{C}_{\mathcal{O}}(V))$ coming from the S_{m+1}-structure on $\mathcal{O}(m)$ and the compatible inner product on V. Then, we can define the *cyclic derivations*, $\mathrm{CDer}(\hat{C}_{\mathcal{O}}(V))$, to be the subspace of $\mathbb{Z}/(m+1)$-invariants in each degree m. Then, a gauge equivalence $e^{\mathrm{ad}\,\xi} : d_{\hat{C}_{\mathcal{O}}(V)} \xrightarrow{\sim} d'_{\hat{C}_{\mathcal{O}}(V)}$ is *cyclic* if ξ is cyclic.

Finally, we explain the main idea of the proof of the claim (without going into full detail). We explain only the second part, since it is more involved. We first form the cyclic deformation complex for the $C(\mathcal{O})$-structure on V structure with all higher operations $C(\mathcal{O})(i), i > 0$ acting as zero, and with 0-ary structure map $\gamma : C(\mathcal{O})(0) = \Pi\mathcal{O}(0)^* \to V$. This complex is $\mathrm{CDer}(\hat{C}_{\mathcal{O}}(V))$, equipped with the differential $\mathrm{ad}\,\xi$, where $\xi \in \mathrm{CDer}(\hat{C}_{\mathcal{O}}(V))$ is the derivation obtained from $\Pi\gamma^* : \Pi V^* \to \mathcal{O}(0)$, by the formula (for $o \in \mathcal{O}(n)$ and $f_1, \ldots, f_n \in \Pi V^*$):

$$\xi(o \otimes_{S_n} (f_1 \otimes \cdots \otimes f_n)) = \sum_{i=1}^{n} (o \circ_i \Pi\gamma^*(f_i)) \otimes_{S_{n-1}} (f_1 \otimes \cdots \otimes \hat{f}_i \otimes \cdots \otimes f_n), \tag{5.10}$$

where $\hat{f}_i$ denotes omitting f_i from the tensor product.

Then, we again show that this complex is quasi-isomorphic to the subcomplex spanned by cochains such that $\mathcal{O}(i)$ acts by multiples of the element

$$\epsilon^{i+1}(x_1, \ldots, x_i) := \epsilon(x_1) \cdots \epsilon(x_i) c', \quad \epsilon(v) := (c', v), \tag{5.11}$$

and hence that the original algebra structure is gauge equivalent to one where all ≥ 1-ary operations are multiples of the operation

$$\phi'(m)(v_1, \ldots, v_i) = \Big(\prod_{j=1}^{i} (c', v_j)\Big) \cdot c'. \tag{5.12}$$

Finally, as before, we can see that this multiple is $(\phi(m)(c, c, \ldots, c), c)$.

The main technical step in the above is showing that the complex is quasi-isomorphic to the subcomplex spanned by cochains as in (5.11). This amounts to a generalization of Lemma 2.17. The main point is that the map s'_i defined in (2.18) can be generalized to this setting, as follows. If $f = o \otimes_{S_{n+1}} (f_1 \otimes \cdots \otimes f_{n+1}) \in \mathcal{O}(n) \otimes_{S_{n+1}} (\Pi V^*)^{\otimes(n+1)}$, viewed as an element of $\mathrm{CDer}(\hat{C}_{\mathcal{O}}(V))$, then $s_n f \in \mathcal{O}(n+1) \otimes_{S_{n+2}} (\Pi V^*)^{\otimes(n+2)}$ is obtained as

$$s_n f = \sum_{j=1}^{n+1} (o \circ_j m_2) \otimes_{S_{n+2}} (f_1 \otimes \cdots \otimes f_{j-1} \otimes \epsilon \otimes f_j \otimes \cdots \otimes f_{n+1}).$$

Then, one obtains a similar result to (2.20), which gives the desired conclusion.

Next, Theorem 3.18 generalizes as follows. Let $\mathcal{O}$ be a cyclic operad with a weakly unital multiplication m_2, and let $\mathcal{G}_{\mathcal{O}}$ be the graph complex constructed from the operad $\mathcal{O}$. This complex is a generalization of Kontsevich's graph homology which has many equivalent definitions in the literature; one definition is via the Feynman transform construction of [GK98]. Namely, consider the naive Det-twisted modular closure $\underline{\mathcal{O}}^1$ of $\mathcal{O}$ by considering all contraction operations to be zero and all parts of $\mathcal{O}$ of genus ≥ 1 to be zero; see, e.g., [CL07, §2] for this notion. Then form the Feynman transform operad $\mathsf{F}\underline{\mathcal{O}}^1$. The 0-ary part $\mathsf{F}\underline{\mathcal{O}}^1((0))$ is the desired graph complex. As before, let $\mathcal{G}_{\mathcal{O},c} \subseteq \mathcal{G}_{\mathcal{O}}$ be the subcomplex spanned by connected nonempty graphs; one has $\mathcal{G}_{\mathcal{O}} \cong \operatorname{Sym} \mathcal{G}_{\mathcal{O},c}$.

Claim 5.13. The complex $\mathcal{G}_{\mathcal{O},c}$ is quasi-isomorphic to the quotient of the subcomplex spanned by graphs with at most one vertex of valence ≥ 2 by the span of line segments with three vertices whose central vertex is labeled by Id.[4]

In other words, $\mathcal{G}_{\mathcal{O},c}$ is quasi-isomorphic to a quotient of the deformation complex of a certain canonical cyclic $C(\mathcal{O})$-algebra (which we think of as a type of curved $((\mathcal{O}')^!)_\infty$-algebra), as follows.[5] Let $V := \mathcal{O}(0)^*$. View V as a $\mathcal{O}$-algebra with all ≥ 1-ary operations trivial, and with 0-ary structure $\mathrm{Id} : \mathcal{O}(0)^* \to \mathcal{O}(0)^*$. Since all higher operations are trivial, any inner product on V is cyclic; we can fix one but it will not really affect anything. Then, the deformation complex of V as a $C(\mathcal{O})$-algebra is

$$\operatorname{CDer}(\hat{C}_{\mathcal{O}}(V)) \cong \bigoplus_{m\geq 0} \mathcal{O}(m) \otimes_{S_{m+1}} \mathcal{O}(0)^{\otimes m+1},$$

equipped with the differential $\operatorname{ad}\xi$ where $\xi \in \operatorname{CDer}(\hat{C}_{\mathcal{O}}(V))$ is the element corresponding to $\mathrm{Id} \in V \otimes \mathcal{O}(0) \cong \operatorname{Hom}(V,V) \subseteq \operatorname{Der}(\hat{C}_{\mathcal{O}}(V))$. By the above, $\mathcal{G}_{\mathcal{O},c}$ is quasi-isomorphic to the quotient of this deformation complex by the subcomplex in arity 2, $\mathrm{Id} \otimes_{S_2} \mathcal{O}(0)^{\otimes 2} \subseteq \bigoplus_{m\geq 0} \mathcal{O}(m) \otimes_{S_{m+1}} \mathcal{O}(0)^{\otimes m+1}$.

The proof of Claim 5.13 is a direct generalization of the proof of Theorem 3.18. First, we generalize the notion of interior vertex. Note that $\mathcal{G}_{\mathcal{O},c}$ is spanned by oriented graphs of the following form: the vertices are ordered, and each m-valent vertex is labeled by an element of $\mathcal{O}(m-1)$. Next, the half-edges are also ordered. Many of these graphs are equivalent: applying a permutation of the half-edges incident to a given vertex is set equal to applying the corresponding permutation to the element of $\mathcal{O}$ labeling that vertex; also, applying any permutation of the half-edges is set equal to multiplying by the sign of that permutation. Finally, applying a permutation to the vertices is the same as multiplying by the sign of that permutation.

Then, an interior vertex of a graph as above is a vertex which has valence ≥ 2 and whose removal does not result in a graph one of whose connected components consists only of univalent vertices and bivalent vertices labeled by elements of $\mathbb{C} \cdot \mathrm{Id} \subseteq \mathcal{O}(1)$. As before, the number of interior vertices defines an increasing filtration. Moreover, if we choose an arbitrary basis $\{\Gamma_i\}$ of graphs of $\mathcal{G}_{\mathcal{O},c}$ (and possibly the empty graph and the graph with

[4]Note that, if $\mathcal{O}(0)$ is even one-dimensional, as in the preceding cases of $\mathscr{C}omm$ or $\mathscr{A}ss$, then these line segments with central vertex labeled by Id are already zero. More generally, the span of these line segments is isomorphic to $\wedge^2\mathcal{O}(0)$, by considering the labels at the univalent vertices.

[5]This interpretation requires that $\mathcal{O}(0)$ be finite-dimensional, as we are assuming. For infinite-dimensional formal $\mathcal{O}(0)$, while the claim above still holds, this interpretation is technically not available.

a vertex and no edges), then the associated graded complex is graded by this basis, where $\mathsf{gr}_{\Gamma_i} \mathcal{G}_{\mathcal{O},c}$ is spanned by graphs whose restriction to interior vertices yields Γ_i.

The main step is to generalize the construction of the contracting homotopy which shows that $\mathsf{gr}_{\Gamma_0} \mathcal{G}_{\mathcal{O},c}$ is acyclic when Γ_0 is a graph containing an edge. To do so, first fix an element $m_0 \in \mathcal{O}(2)$ such that $m_2 \circ_1 m_0 = m_2 \circ_2 m_0 = \mathrm{Id} \in \mathcal{O}(1)$. Fix a half edge h of Γ_0 based at a vertex v. Let us choose v to be the last vertex in the ordering of vertices, and h to be the last half-edge in the ordering of half-edges based at v. Suppose v is m-valent, and let the label of v be $o_v \in \mathcal{O}(m-1)$. Call the half-edges of v, in order, $h_1, \ldots, h_m$, with $h = h_m$. The contracting homotopy s then acts on any graph $\Gamma \in \mathsf{gr}_{\Gamma_0} \mathcal{G}_{\mathcal{O},c}$ by first adding a new half-edge h_{m+1} to v (last in the ordering at v). Then, the label o_v is replaced by $o_v \circ_m m_2$, where $m_2 \in \mathcal{O}(2)$ is the weakly unital multiplication. Finally, one adds a new univalent vertex y to Γ (which becomes the last vertex), labels it by m_0, and attaches it to h_{m+1}. It is then straightforward to verify that $(sd + ds)\Gamma = \Gamma$, and hence $\mathsf{gr}_{\Gamma_0} \mathcal{G}_{\mathcal{O},c}$, is acyclic.

Finally, we explain the appearance of the quotient by line segments with three vertices whose central vertex is labeled by Id. Namely, these are exactly the graphs with a single vertex of valence ≥ 2 which, nonetheless, have no interior vertices. By a generalization of the arguments in the proof of Theorem 3.18, $\mathsf{gr}_{\mathrm{pt}} \mathcal{G}_{\mathcal{O},c}$ is quasi-isomorphic to the quotient of the subcomplex spanned by star-shaped graphs with a single vertex of valence ≥ 2 by the span of these graphs. On the other hand, $\mathsf{gr}_{\emptyset} \mathcal{G}_{\mathcal{O},c}$ is quasi-isomorphic to the subcomplex spanned by graphs with two vertices, each univalent. Thus, the second page of the spectral sequence for the interior vertex filtration yields, in degree one, the claimed quotient complex modulo the graphs with only two vertices, and in degree zero, the span of graphs with only two vertices and one edge. The spectral sequence collapses at the third page to the homology of the whole quotient subcomplex stated in Claim 5.13, which proves the result. We omit further details.

Finally, one can deduce an unstable version of Claim 5.13, analogous to Theorem 4.3:

Claim 5.14. The composition

$$\mathrm{CE}_\bullet(\mathrm{CDer}^0(\hat{C}_{\mathcal{O}}(V))) \to \mathrm{CE}_\bullet(\mathrm{CDer}(\hat{C}_{\mathcal{O}}(V))) \to \mathrm{CE}_\bullet(\mathrm{CDer}(\hat{C}_{\mathcal{O}}(V \oplus W))) \tag{5.15}$$

is zero on homology for any uncurved cyclic $C(\mathcal{O})$-algebra V, where W is an inner product space with $W_0 \neq 0$.

We omit the proof, which is obtained by combining the preceding material with §4.

6. Examples and further comments

In this section we provide some remarks and examples regarding the material of the previous section.

6.1. **Generalization to the modular case.**

Here we briefly sketch how the results of the previous section can be extended from the setting of cyclic operads to that of modular operads.

An analogous result to Claim 5.13 holds for the *twisted* version of the $\mathcal{O}$-graph complex which corresponds to the Feynman transform $\mathsf{F}\underline{\mathcal{O}}((0))$ (as opposed to $\mathsf{F}\underline{\mathcal{O}}^1((0)))$ which is the

usual version of the graph complex.[6] For example, in the commutative and ribbon graph case the difference between two types of graph complex lies in a different notion of orientation: a twisted orientation corresponds to ordering edges of a graph, as opposed to ordering the half edges and vertices.

More generally, one can replace cyclic operads $\mathcal{O}$ by arbitrary (twisted) modular operads. Then, if $\mathcal{O}$ admits a weakly unital multiplication, one still obtains in the same manner the result that the graph complex for $\mathcal{O}$ is quasi-isomorphic to the subcomplex defined analogously to the above.

Similarly, we can generalize all the constructions of the preceding section from the cyclic to the modular setting. Let $\mathcal{O}$ be an arbitrary (twisted) modular operad $\mathcal{O}$. In this case, one replaces $C(\mathcal{O})$ (used in the cyclic case above) by the Feynman transform (twisted) modular operad, $\mathsf{F}\mathcal{O}$, and modular algebras over this operad are then thought of as curved algebras. The constructions, results (Claims 5.8.(ii) and 5.14), and proofs carry over to this setting. In the special case where one has a cyclic operad and considers it Det-twisted modular using the naive Det-twisted modular closure, one recovers the above results. Another example concerns so-called quantum A_∞-algebras: Let $\mathcal{O}'$ be the Det-twisted modular closure of $\mathscr{A}ss$ such that $\mathsf{F}\mathcal{O}'$-algebras are so-called quantum A_∞-algebras (see for instance [CL10, Example 5.2] for an explanation of this notion). Then, if we let $\mathcal{O} = \mathcal{O}' \oplus \mathbb{C}[0]$ be the unital version (adding a 0-ary operation providing a unit for the associative multiplication), then $\mathsf{F}\mathcal{O}$ defines a notion of curved quantum A_∞-algebras. As in the cyclic A_∞ case, one sees that nontrivially curved quantum A_∞-algebra structures on a fixed vector space are gauge equivalent to those where all the operations of positive arity are of the form (5.9), and in particular all land in a fixed one-dimensional vector space. We refrain from making precise statements.

6.2. **Examples.** Here we provide examples of the preceding constructions for Poisson, Gerstenhaber, BV, and permutation (or pre-Lie) algebras.

Example 6.1. Consider the case of Poisson algebras. It is natural to consider unital Poisson algebras, where here a unit f is an element satisfying $\{1, f\} = \{f, 1\} = 0$ for all f, whereas $1 \cdot f = f \cdot 1 = f$. Let $\mathcal{O}$ be the operad governing unital Poisson algebras, i.e., $\mathcal{O} = u\mathscr{P}oiss = \mathscr{P}oiss \oplus \mathbb{C}[0]$ where $\mathscr{P}oiss$ is the Poisson operad and $\mathbb{C}[0]$ is the one-dimensional vector space concentrated in degree zero, which has zero compositions with the bracket of $\mathcal{O}$, and composition $m_2(m_0 \otimes \mathrm{Id}) = \mathrm{Id} = m_2(\mathrm{Id} \otimes m_0)$ with the commutative multiplication, i.e., $\mathscr{P}oiss \supset u\mathscr{C}omm$. Then, by the above, one can think of $C(\mathcal{O})$-algebras as curved Poisson-infinity algebras (since Poisson is Koszul self-dual, as in the associative case), and it follows from Claim 5.8.(i) that nontrivially curved Poisson-infinity structures on a vector space are all gauge equivalent.

Similarly, the Gerstenhaber operad $\mathscr{G}erst$ is Koszul and dual to its suspension $\mathfrak{s}\mathscr{G}erst$; here the suspension is defined by tensoring by the endomorphism operad of the one-dimensional odd vector space (as a $\mathbb{Z}/2$-graded $\mathbb{S}$-module, this means that one applies Π to the even-ary part). As before, one can consider unital $\mathscr{G}erst$ algebras, $u\mathscr{G}erst$, and similarly its suspension $\mathfrak{s}(u\mathscr{G}erst)$. Let $\mathcal{O}$ be this latter operad, which we can also think of as $u\mathscr{G}erst^!$. Here, the unit is odd. We define curved $\mathscr{G}erst_\infty$ algebras as algebras over $C(\mathcal{O})$, where now the

[6] We note that the operads $\mathsf{F}\underline{\mathcal{O}}^d((n))$ are not, strictly speaking, modular operads in the sense of [GK98], since the stability condition is not satisfied; however the construction of the Feynman transform still makes sense and this causes us no trouble. We will ignore this point henceforth.

curvature is odd, rather than even. From Claim 5.8.(i), we deduce that any two nontrivially curved Gerstenhaber-infinity algebra structures on the same graded vector space are gauge equivalent.

In the case of the Poisson operad (but not for the Gerstenhaber operad), in fact $\mathscr{P}oiss$ is cyclic, and hence one obtains the notion of cyclic curved Poisson-infinity algebras. Claim 5.8.(ii) then implies that all nontrivially curved cyclic Poisson-infinity structures on a vector space are gauge equivalent to those for which all operations of positive arity are of the form (5.9), and in particular all land in a fixed one-dimensional vector space.

Moreover, in fact one can compute the associated (unital Poisson) graph homology. By Claim 5.13, the associated graph complex is quasi-isomorphic to the subcomplex spanned by graphs whose vertices are all univalent except for at most one. Since the Poisson operad is the associated graded operad of the associative operad, and in particular has the same $\mathbb{S}$-module structure, one sees that this subcomplex has zero differential, and is spanned by the star-shaped graphs with central vertex of odd valence. That is, the graph homology for the unital Poisson operad is isomorphic to the homology of the graph complex $\widetilde{\mathcal{G}}_r$ for the unital associative operad.

Example 6.2. The BV operad $\mathscr{BV}$ [Get94] is the homology operad of the operad of framed little discs; it is known to be cyclic. Its algebras, called $\mathscr{BV}$-algebras, are Gerstenhaber algebras together with an odd operator Δ which is a differential operator of second order with respect to the commutative multiplication and a derivation of the odd Lie bracket. Let $\overline{\mathscr{BV}}$, as before, denote the augmentation ideal of the operad $\mathscr{BV}$. The $C(\overline{\mathscr{BV}})$-graph complex computes, according to [Gia10], the homology of the classifying space of diffeomorphism groups of 3-dimensional oriented handlebodies. Although the operad $\mathscr{BV}$ is not defined by (homogeneous) quadratic relations, in [GCTV09] it was shown to be Koszul in a more general sense, and its Koszul dual, $\mathscr{BV}^!$ was described, and shown to be quasi-isomorphic to $C(\overline{\mathscr{BV}})$.

A unital $\mathscr{BV}$-algebra is a $\mathscr{BV}$-algebra with a unit with respect to the commutative multiplication and such that the value of Δ on the unit is zero.

Consider the operad $\mathcal{O} = u\mathscr{BV}$ governing unital $\mathscr{BV}$-algebras; then we can view $C(\mathcal{O})$ as the operad governing curved $\mathscr{BV}^!_\infty$-algebras. It follows that all nontrivially curved $\mathscr{BV}^!_\infty$-algebras are gauge equivalent. Furthermore, according to Claim 5.8.(ii) all nontrivially curved cyclic $\mathscr{BV}^!_\infty$-algebras are gauge equivalent to those for which all operations of positive arity are of the form (5.9).

Finally, by Claim 5.13, the graph complex for $\mathcal{O}$ is quasi-isomorphic to the subcomplex with at most one vertex of valence ≥ 2.

Example 6.3. Consider the case of (right) pre-Lie algebras, i.e., those with a single operation $\star$ satisfying the relation

$$x \star (y \star z) - (x \star y) \star z = x \star (z \star y) - (x \star z) \star y. \tag{6.4}$$

For such algebras, it makes perfect sense to define the notion of a unit, 1, such that

$$1 \star x = x = x \star 1. \tag{6.5}$$

We thus obtain an operad $\mathcal{O} = u\mathscr{P}re\text{-}\mathscr{L}ie = \mathscr{P}re\text{-}\mathscr{L}ie \oplus \mathbb{C}[0]$ governing *unital* pre-Lie algebras, where now the compositions with $\mathbb{C}[0]$ are given by (6.5), or more precisely, $m_2(m_0 \otimes \mathrm{Id}) = \mathrm{Id} = m_2(\mathrm{Id} \otimes m_0)$ where $m_0 = 1 \in \mathbb{C}[0]$ and $m_2 \in \mathscr{P}re\text{-}\mathscr{L}ie[2]$ is the multiplication operation

$\star$. By the above procedure, we then obtain a type of curved algebra, namely algebras over $C(\mathcal{O})$. We will call these *curved (right) $\mathscr{P}erm_\infty$-algebras*, since the operad (right) $\mathscr{P}erm$ is Koszul dual to (right) $\mathscr{P}re\text{-}\mathscr{L}ie$. In particular, curved $\mathscr{P}erm_\infty$ algebras with zero curvature ($m_0 = 0$, i.e., the corresponding derivation of $C_{\mathcal{O}}(V)$ has zero constant term) are the same as ordinary $\mathscr{P}erm_\infty$ algebras. Let us recall here that ordinary permutation algebras are algebras with an operation $\circ$ satisfying the relation

$$x \circ (z \circ y) = x \circ (y \circ z) = (x \circ y) \circ z, \tag{6.6}$$

i.e., associative algebras additionally satisfying the first equality above. The operad $\mathscr{P}erm$ is the one whose algebras are permutation algebras.

From the above results, we deduce that any two curved $\mathscr{P}erm_\infty$-algebra structures on V with nonzero curvature are gauge-equivalent.

Note that, strictly speaking the operad $\mathscr{P}re\text{-}\mathscr{L}ie$ is not cyclic, it is anticyclic [GK95, p. 9]; however we can ignore this difference; any anticyclic operad gives rise to a cyclic one via the operadic suspension. Therefore, we obtain the corresponding result on the classification of nontrivially curved (anti)cyclic $\mathscr{P}erm_\infty$-algebras, and on $u\mathscr{P}re\text{-}\mathscr{L}ie$-graph homology.

6.3. **On the cobar construction of unital operads.** Finally, we remark that there is a certain subtlety associated with taking cobar-constructions of unital operads (such as an operad with a weakly unital multiplication), as we do. Let $\mathcal{O}$ be such an operad. Then it is easy to see that for any $n \geq 0$ the complex $C\mathcal{O}(n)$ is contractible. However it does not follow that any $C\mathcal{O}$-algebra is gauge equivalent to a trivial one. Indeed, one can take $\mathcal{O}$ to be the operad $u\mathscr{A}ss$ or $u\mathscr{C}omm$ governing unital associative or unital commutative algebras; then $C\mathcal{O}$-algebras (on a graded vector space V with zero differential) will be curved A_∞- or L_∞-algebras, respectively, and there is no reason for an arbitrary curved A_∞- or L_∞-algebra to be gauge equivalent to a trivial one. In fact, even if we let $\mathcal{O}$ be the operad $\mathscr{A}ss$ or $\mathscr{C}omm$, and take the bar construction of it (i.e., of the whole unital operad, rather than just the augmentation ideal as one usually does in these cases), we again obtain that $C\mathcal{O}$-algebras will be ordinary A_∞- or L_∞-algebras, with the usual (nontrivial) gauge equivalence relation, even though $C\mathcal{O}$ remains acyclic.

The explanation of this apparent paradox is that the operad $C\mathcal{O}$ is *not cofibrant*; see [BM03] for this notion. An algebra over $C\mathcal{O}$ is a map from $C\mathcal{O}$ to an endomorphism operad of a dg vector space and this map is not necessarily homotopic to zero even though $C\mathcal{O}$ is acyclic. A similar phenomenon occurs when considering a cobar-construction $(T\Pi V^*, d)$ for a unital associative algebra V; dg maps from $T\Pi V^*$ to the field $\mathbb{C}$ are the Maurer-Cartan elements in V, i.e. the odd elements $v \in V$ for which $dv + v^2 = 0$. Such elements need not be gauge equivalent to zero despite $(T\Pi V^*, d)$ being acyclic; again, precisely because $(T\Pi V^*, d)$ is not a cofibrant dga.

Furthermore, given a cyclic operad $\mathcal{O}$ as above we can form its Det^d-modular closure $\overline{\mathcal{O}}^d$ and its naive Det^d-modular closure $\underline{\mathcal{O}}^d$. Then the same reasoning shows that the complexes $\mathsf{F}\overline{\mathcal{O}}^d((n))$ and $\mathsf{F}\underline{\mathcal{O}}^d((n))$ are acyclic for $n > 0$; here $\mathsf{F}\mathcal{O}$ is the Feynman transform of $\mathcal{O}$.

Next, the complexes $\mathsf{F}\underline{u\mathscr{C}omm}^1((0))$ and $\mathsf{F}\underline{u\mathscr{A}ss}^1((0))$ are nothing but our graph complexes $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{G}}_r$. For $n > 0$, $\mathsf{F}\underline{u\mathscr{C}omm}^1((n))$ and $\mathsf{F}\underline{u\mathscr{A}ss}^1((n))$ are similar except they are spanned by graphs which are additionally equipped with n external labeled edges (legs) which are not allowed to be contracted. When legs are present, these graph complexes are therefore acyclic.

However, the vacuum (legless) part $\mathsf{F}\overline{\mathcal{O}}^d((n))$ and $\mathsf{F}\underline{\mathcal{O}}^d((n))$ of the Feynman transform need not be acyclic, as our results demonstrate.

References

[BM03] C. Berger and I. Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. **78** (2003), no. 4, 805–831.

[CL07] J. Chuang and A. Lazarev, *Dual Feynman transform for modular operads*, Commun. Number Theory Phys. **1** (2007), no. 4, 605–649.

[CL10] ———, *Feynman diagrams and minimal models for operadic algebras*, J. Lond. Math. Soc. (2) **81** (2010), no. 2, 317–337.

[Cos09] K. Costello, *The partition function of a topological field theory*, J. Topol. **2** (2009), no. 4, 779–822.

[GCTV09] I. Galvez-Carrillo, A. Tonks, and B. Vallette, *Homotopy Batalin-Vilkovisky algebras*, arXiv:0907.2246, 2009.

[Get94] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. **159** (1994), no. 2, 265–285.

[Gia10] J. Giansiracusa, *The framed little 2-discs operad and diffeomorphisms of handlebodies*, arXiv:1008.3272, 2010.

[GJ90] E. Getzler and J. D. S. Jones, *A_∞-algebras and the cyclic bar complex*, Illinois J. Math. **34** (1990), no. 2, 256–283.

[GJ94] E. Getzler and J.D.S. Jones, *Operads, homotopy algebra and iterated integrals*, arXiv:hep-th/9403055, 1994.

[GK94] V. Ginzburg and M. Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1994), no. 1, 203–272, arXiv:0709.1228.

[GK95] E. Getzler and M. M. Kapranov, *Cyclic operads and cyclic homology*, Geometry, topology, & physics (Cambridge, MA), Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, 1995, pp. 167–201.

[GK98] ———, *Modular operads*, Compositio Math. **110** (1998), no. 1, 65–126, arXiv:dg-ga/9408003.

[GS73] V. Guillemin and S. Shnider, *Some stable results on the cohomology of the classical infinite-dimensional Lie algebras*, Trans. Amer. Math. Soc. **179** (1973), 275–280.

[Ham06] A. Hamilton, *A super-analogue of Kontsevich's theorem on graph homology*, Lett. Math. Phys. **76** (2006), no. 1, 37–55.

[Ham09] ———, *Classes on compactifications of the moduli space of curves through solutions to the quantum master equation*, Lett. Math. Phys. **89** (2009), no. 2, 115–130.

[HL08] A. Hamilton and A. Lazarev, *Characteristic classes of A_∞-algebras*, J. Homotopy Relat. Struct. **3** (2008), no. 1, 65–111.

[HL09] ———, *Cohomology theories for homotopy algebras and noncommutative geometry*, Algebr. Geom. Topol. **9** (2009), no. 3, 1503–1583.

[HM10] J. Hirsh and J. Millès, *Curved Koszul duality theory*, http://math.unice.fr/∼jmilles/math/Research.html, 2010.

[How95] R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, The Schur lectures (1992) (Tel Aviv), Israel Math. Conf. Proc., vol. 8, Bar-Ilan Univ., Ramat Gan, 1995, pp. 1–182.

[HVZ07] E. Harrelson, A. Voronov, and J. Javier Zuniga, *Open-closed moduli spaces and related algebraic structures*, arXiv:0709.3874, 2007.

[Igu04] K. Igusa, *Graph cohomology and Kontsevich cycles*, Topology **43** (2004), no. 6, 1469–1510.

[KLN10] B. Keller, W. Lowen, and P. Nicolás, *On the (non)vanishing of some "derived" categories of curved dg algebras*, J. Pure Appl. Algebra **214** (2010), no. 7, 1271–1284.

[Kon93] M. Kontsevich, *Formal (non)commutative symplectic geometry*, The Gelfand Mathematial Seminars, 1990–1992, Birkhäuser Boston, MA, 1993, pp. 173–182.

[Kon94] ______, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II (Paris, 1992), Progr. Math., vol. 120, Birkhäuser, Basel, 1994, pp. 97–121.

[Lef42] S. Lefschetz, *Algebraic Topology*, American Mathematical Society Colloquium Publications, v. 27, American Mathematical Society, New York, 1942.

[Nic08] P. Nicolás, *The bar derived category of a curved dg algebra*, J. Pure Appl. Algebra **212** (2008), no. 12, 2633–2659.

[Pos93] L. E. Positsel′skiĭ, *Nonhomogeneous quadratic duality and curvature*, Funktsional. Anal. i Prilozhen. **27** (1993), no. 3, 57–66, 96.

[Ser01] A. Sergeev, *An analog of the classical invariant theory for Lie superalgebras. I, II*, Michigan Math. J. **49** (2001), no. 1, 113–146, 147–168.

[SZ96] A. Sen and B. Zwiebach, *Background independent algebraic structures in closed string field theory*, Comm. Math. Phys. **177** (1996), no. 2, 305–326.

[Wey39] H. Weyl, *The Classical Groups. Their Invariants and Representations*, Princeton University Press, Princeton, N.J., 1939.